Necessary Conditions for Mode Interactions in Parametrically Excited Waves

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We study the spatial and temporal structure of nonlinear states formed by parametrically excited waves on a fluid surface (Faraday instability), in a highly dissipative regime. Short-time dynamics reveal that 3-wave interactions between different spatial modes are *only* observed when the modes' peak values occur simultaneously. The temporal structure of each mode is functionally described by the Hill's equation and is unaffected by which nonlinear interaction is dominant.

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In a nonlinear system, where a number of linearly unstable modes are concurrently excited, nonlinear interactions wholly determine the structure of the resulting nonlinear state. We consider the conditions for two distinct parametrically excited waves to couple to a mode that is linearly damped. We find that temporally local phase matching between the driven waves is a necessary condition for this 3-wave coupling to occur. The simplicity of this condition suggests that it may be generally applicable in governing nonlinear interactions between excited modes in a wide class of dissipative nonlinear systems [1].

The Faraday system, in which a thin layer of fluid is subjected to spatially uniform vertical vibration, is commonly used as a model system when investigating pattern formation [2]. When the vibration is purely sinusoidal, surface waves are parametrically excited beyond a critical acceleration. The frequency of these waves is half of the driving frequency, and the wave number, k, of this excited mode is selected via the dispersion relation. When driving the system with multiple frequencies [3], a number of different modes can be concurrently excited. Under such driving, a rich variety of complex nonlinear (standing wave) patterns has been observed. These include quasicrystalline patterns and a variety of different superlattice states, when nonlinear interactions between modes with different length scales [3–5] take place.

In the work described here, we drive the system with two commensurate frequencies, where the imposed acceleration has the form

$$a(t) = a_{m_1} \cos(m_1 \omega t) + a_{m_2} \cos(m_2 \omega t + \phi) \qquad (1)$$

where m_1 and m_2 are two coprime integers, ϕ is the relative driving phase, and ω is the common frequency. Our experimental system is based on a design by Kityk *et al.* [6], which uses light absorption to measure the height of the surface separating two immiscible fluids. We used a circular cell, of 150 mm diameter and 11 mm height, that was filled with silicone oil (DC200/20, $\nu = 18$ cSt) of depth 9.9 mm and a 1.1 mm deep solution of water, Glycerol and Janus Green B dye of viscosity $\nu = 5$ cSt. The refractive index, n, of this solution was index-matched to that of the silicone oil, with an accuracy of $\Delta n = 0.005$. The fluid temperature was controlled to an accuracy of 0.02 °C via IR lamps. The fluids were bounded by an optical window from above and a mirrored surface from below. A 150 mm diameter beam of collimated monochromatic (630 nm) light traversed both fluids and was imaged by a CCD camera. As the index-matching eliminated light deflection at the fluid surfaces, the intensity of each image point was wholly determined by the beam's exponential absorption through the dyed layer. Thus, the logarithm of the intensity yielded a precise measurement of the surface height at each spatial point in the system. The camera was triggered at different times, $\tau = t \bmod (2\pi/\omega)$, to measure the spatial and temporal evolution of the surface amplitude. The temporal dependence of the amplitude, $A_k(\tau)$, of each spatial mode was obtained by spatial demodulation of each image for each wave number, k.

Most of this work was performed with the frequency ratio $m_1{:}m_2 = 2{:}3$, with similar results obtained using other frequency ratios. For the $m_1{:}m_2 = 2{:}3$ ratio modes with wave numbers $k_{2\omega}$ and $k_{3\omega}$, corresponding to the driving frequencies 2ω and 3ω, can be directly excited. A phase diagram of the system's response to this forcing is shown in Fig. 1(a). The flat state is a featureless state where the system is below the critical acceleration for each frequency. The right (top) border of the flat state denotes the critical acceleration where the mode $k_{2\omega}$ ($k_{3\omega}$), corresponding to the 2ω (3ω) driving component, loses stability and forms a square pattern, 2ω-Squares (3ω-Squares). An example of a square pattern and its spatial spectrum is shown in Fig. 1(b) top. One prominent feature in the phase diagram is that the stability lines are nearly straight [7], indicating that the stability of each mode depends almost entirely on its respective forcing component.

The two stability lines coincide at a codimension-2 point, where both driven modes are simultaneously unstable. In some regions of phase space beyond this point, superlattices, called 2-Mode Superlattice states (2MS) [5], are observed [Fig. 1(b)]. The 2MS states are not simple superpositions of the two excited modes $k_{2\omega}$ and $k_{3\omega}$, but are formed by the three-wave interaction that couples these

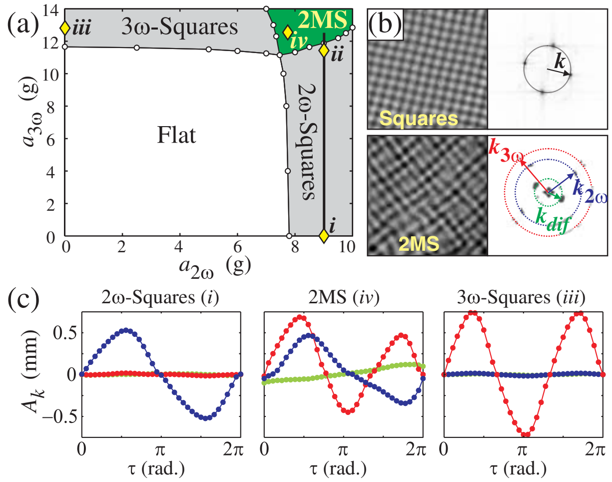

FIG. 1 (color online). (a) Phase diagram of the system for frequency ratio $m_1{:}m_2 = 2{:}3$, $\phi = 0$, and $\omega/2\pi = 11$ Hz. The 3ω-Squares and the 2ω-Squares phases are, respectively, governed by $k_{3\omega}$ and $k_{2\omega}$. (b [top]) Image (right) and spatial spectrum (left) of a typical square state. The 2-Mode superlattice (2MS) state (b [bottom]) is formed by a three-wave interaction between $k_{2\omega}$ and $k_{3\omega}$ and the linearly damped mode $\vec{k}_{\rm dif} = \vec{k}_{3\omega} - \vec{k}_{2\omega}$. Circles of radii k are drawn. (c) Examples of the temporal response of amplitudes, A_k, in each phase for $k_{3\omega}$ (medium shade/red line), $k_{2\omega}$ (dark shade/blue line), and $k_{\rm dif}$ (light shade/green line), whose respective fundamental frequencies are $\frac{3}{2}\omega$, ω, and $\frac{1}{2}\omega$. Only in the 2MS state are more than a single A_k excited. $0 < \tau \leq t \bmod (2\pi/\omega)$.

modes to the "difference" mode, $k_{\rm dif}$, a linearly damped mode that is excited by this nonlinear interaction. Its wavelength and frequency are determined by $\vec{k}_{\rm dif} = \vec{k}_{3\omega} - \vec{k}_{2\omega}$ and $\omega_{\rm dif} = \frac{3}{2}\omega - \frac{1}{2}\omega$. Although studied theoretically using broken temporal symmetries [8], the conditions conducive to this interaction are not well understood.

In Fig. 1(c), we present examples of the temporal behavior of the amplitudes, $A_k(\tau)$, of each of the spatial modes composing the different nonlinear states. While the overall temporal response of each $A_k(\tau)$ corresponds to the *fundamental* frequencies of the excited linear modes $k_{3\omega}$, $k_{2\omega}$, and $k_{\rm dif}$, each $A_k(\tau)$ differs significantly from a pure sinusoidal oscillation throughout the entire phase diagram. We will show that these wave forms influence the nature of the 3-wave interaction that forms the 2MS state.

We first consider how the $A_k(\tau)$, composing the square states, evolve throughout the phase diagram. To this end, we studied the evolution of $A_k(\tau)$ for paths in phase space where one of the driving accelerations is varied, while the other is fixed. An example of the evolution of $A_{k_{2\omega}}(\tau)$ along the path connecting points *i*–*ii* in Fig. 1(a) is presented in Fig. 2. As expected [9], at $a_{3\omega} = 0$, the spectrum [Fig. 2(b) (top)] of $A_{k_{2\omega}}(\tau)$ contains only odd multiples of the fundamental harmonic, ω, with amplitudes similar to recent single-frequency measurements [10]. For $a_{3\omega} > 0$, the

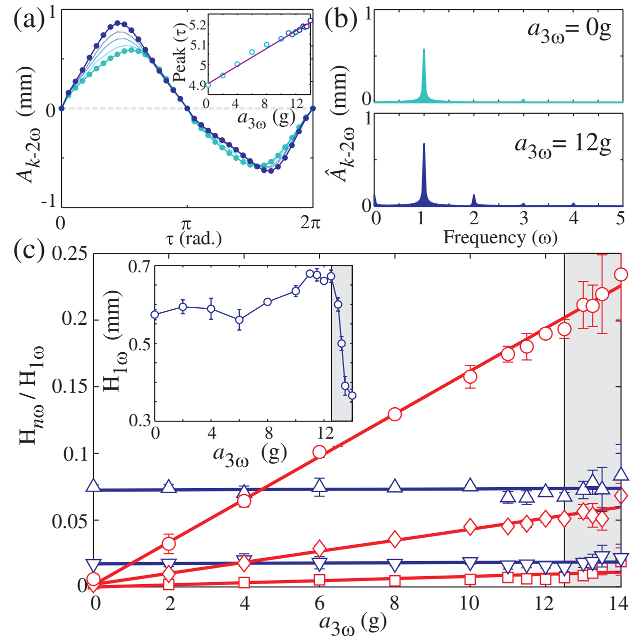

FIG. 2 (color online). Evolution of $A_{k_{2\omega}}(\tau)$ with $a_{3\omega}$. (a) $A_{k_{2\omega}}(\tau)$ along the path *i*–*ii* in Fig. 1(a) ($a_{2\omega} = 9.25g$), where $(0 < \tau < 2\pi/\omega)$. $A_{k_{2\omega}}(\tau)$ evolves from single driving ($a_{3\omega} = 0g$) at point *i* (light/cyan line) to point *ii* (dark/blue line), adjacent to the 2MS phase. (inset) The location in τ of the peak values of $A_{k_{2\omega}}(\tau)$ with $a_{3\omega}$. The temporal spectrum of $A_{k_{2\omega}}(\tau)$ at $a_{3\omega} = 0g$ (b [top]) is composed of the fundamental harmonic at ω and additional harmonics at 3ω, 5ω. (b [bottom]) The spectrum at point *ii*; $A_{k_{2\omega}}(\tau)$ contains new harmonics at 0, 2ω, 4ω for all $a_{3\omega} > 0g$. (c) The amplitudes of all $n > 1$ harmonics, $H_{n\omega}$, are slaved to the amplitude $H_{1\omega}$ of the $n = 1$ fundamental mode. $H_{n\omega}/H_{1\omega}$ are constant for odd values of n (dark/blue line: 3ω-△, 5ω − ▽) and linearly dependent on $a_{3\omega}$ for even values of n (light/red line: 2ω-○, 4ω-◇, 6ω-□). This functional dependence continues in the (shaded) 2MS phase, despite dramatic changes in $H_{1\omega}$ (inset).

spectrum ([Fig. 2(b)] of $A_{k_{2\omega}}(\tau)$ acquires new harmonics at 0 (DC), 2ω, 4ω. These cause a linear shift in $a_{3\omega}$ of the locations of the peak values of the waveform [Fig. 2(a) (inset)].

This spectral structure is expected in the linearized problem with low dissipation. In this limit, the temporal response of the modes is described by Hill's equation [11] for which analytic predictions [12] exist for the dependence of the harmonics on both the accelerations and driving phase, ϕ. This analysis predicts new resonances of $m_2 \pm \frac{1}{2}m_1$, $m_2 \pm \frac{3}{2}m_1$, etc., which result from the interaction between the $A_{k_{2\omega}}(\tau)$ and the $a_{3\omega}$ driving. The amplitudes of these new resonances are expected to be bilinear functions of both $a_{3\omega}$ and the amplitude, $H_{1\omega}$, of the fundamental harmonic, with the amplitudes of the original harmonics proportional to $H_{1\omega}$ but unaffected by $a_{3\omega}$.

Despite the fact that our system is highly dissipative, these predictions might be expected to be valid in the *perturbative* range, where linearization of the equations is valid. Surprisingly, the predicted functional dependence of both sets of harmonics is observed over the *entire* accessible range of $a_{3\omega} \geq 0$. This is demonstrated in Fig. 2, where the development of the higher harmonics,$H_{n\omega}$, normalized by $H_{1\omega}$, is presented for $0 \leq a_{3\omega} < 14g$. As predicted in the perturbative limit, the relative amplitudes of the harmonics that correspond to the 2ω forcing (3ω, 5ω, $\ldots$) remain constant, while each of the new harmonics (2ω, 4ω, 6ω) grows linearly with $a_{3\omega}$, with the linear coefficient independent of the value of $a_{2\omega}$. In addition, the phases of all the harmonics are unaffected by $a_{3\omega}$. Analogous behavior is observed in phase space regions where $k_{3\omega}$ is excited.

$H_{n\omega}$ are all slaved to the fundamental harmonic, although $H_{1\omega}$ itself [Fig. 2(c) (inset)], is a highly nonlinear function of both $a_{2\omega}$ ([10]) and $a_{3\omega}$ throughout the entire square-dominated region of phase space. Surprisingly, this behavior carries over into the 2MS phase, where no appreciable changes in the functional dependence of any of the harmonics are observed despite large variations of $H_{1\omega}$ with $a_{3\omega}$. This slaved dependence is well beyond the linearized region where Hill's equation could be expected to be valid and, *a posteriori*, justifies a basic assumption of previous (weakly nonlinear) theoretical treatments [11] that harmonics beyond the fundamental could be ignored.

We now consider the conditions for which nonlinear waves will couple to form 2MS. As shown in Fig. 1(a), away from the codimension 2 point, the threshold for the 2MS state is well above the linear threshold of both $k_{2\omega}$ and $k_{3\omega}$. Thus, linear excitation of both modes is not a sufficient condition for exciting the 2MS state, and additional conditions are necessary for the interaction to occur.

Let us now consider the relative temporal locations of the peak values of the A_k. We have seen that [Fig. 2(a) (inset)] the addition of the second frequency leads to a large ($\sim 20°$) shift in the location of the peak value of $A_{k_{2\omega}}(\tau)$. Looking now at the 2MS state, we observe that the location, in τ, at which $A_{k_{\rm dif}}(\tau)$ attains its maximal amplitude closely corresponds to the locations of peaks of the two externally driven waveforms, $A_{k_{2\omega}}(\tau)$ and $A_{k_{3\omega}}(\tau)$ [Fig. 3(a)]. This occurs throughout the entire 2MS phase, where the differences in locations of peaks of both $|A_{k_{2\omega}}(\tau)|$ and $|A_{k_{3\omega}}(\tau)|$ are tightly centered around about 5° [Fig. 3(c)]. Similar peak alignment occurs when 2MS states are generated while driving the system at frequency ratios of 4:5 [Fig. 3(b)] and 5:8. In addition, in the region where $A_{k_{\rm dif}}(\tau)$ attains its maximum amplitude, a local dip occurs in the peak amplitudes of the externally excited waveforms, suggesting energy transfer from the externally excited modes to the difference mode. Both the temporal alignment of the waveform peaks and the apparent energy transfer when this alignment occurs suggest that temporal *locality* of the mode interactions is a *necessary* condition for the formation of the 2MS state.

If the alignment of local peaks is a necessary condition for 3-wave interactions, we should be able to enhance or

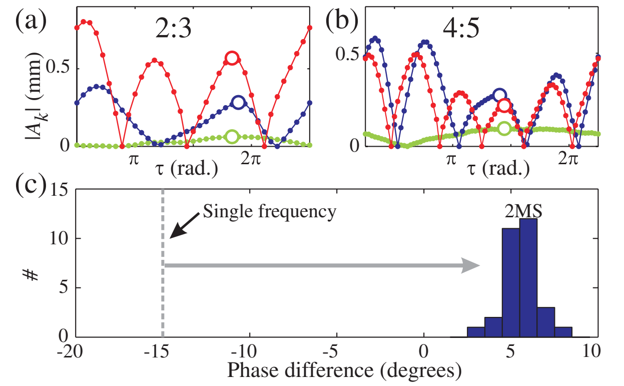

FIG. 3 (color online). $|A_k(\tau)|$ for $k_{2\omega}$ (dark shade/blue line), $k_{3\omega}$ (medium shade/red line) and $k_{\rm dif}$ (light shade/green line) in 2MS state for frequency ratios 2:3 (a) and 4:5 (b). When $|A_{k_{\rm dif}}(\tau)|$ is at a maximum, the peaks of all of the modes are aligned (symbols). (c) Histogram of the maximal difference in τ of the peak locations of these 3 modes throughout the entire 2MS phase. The peaks are aligned to within $\sim 5°$ (dashed line). Difference in τ of the peak locations of $A_{k_{2\omega}}(\tau)$ and $A_{k_{3\omega}}(\tau)$ for single (2ω) frequency forcing.

suppress these nonlinear interactions by varying the relative phases of the two forcing accelerations. To this end, we fixed $a_{2\omega}$ and $a_{3\omega}$ at a point [point *iv* in the center of the 2MS phase in Fig. 1(a)], and varied the relative driving phase, ϕ, in Eq. (1). As shown in Fig. 4, the 2MS state is *only* observed when alignment occurs between local peaks of $|A_{k_{2\omega}}(\tau)|$ and $|A_{k_{3\omega}}(\tau)|$. Once the peaks are too far apart (e.g. $60° < \phi < 140°$), the 2MS state is lost and, instead, a 2ω-Square pattern is excited. For $140° \leq \phi \leq 180°$, the peak values are aligned for a different sets of peaks (τ values) and, once again, the 2MS state appears.

Over the entire $0° \leq \phi \leq 180°$ range, the amplitudes of all temporal harmonics remain [Fig. 4(c)] locked to the amplitude of the fundamental harmonic, despite large changes in its amplitude [Fig. 4(d)]. As in the square regime, the dependence of all of the harmonics is still that predicted by Hill's equation. These observations suggest that the excitation of the 2MS state is not due to nonlinear phase locking [13], but is, instead, the result of the phase conditions being *conducive* to the interaction. We, therefore, suggest that the local alignment of peak amplitudes is a necessary condition for mode interactions in this system.

Three-wave interactions are the most basic nonlinear interaction, when allowed by symmetry. We have shown that the linear instability of the dominant interacting waves is not a sufficient condition for 3-wave interactions, and that, for the interaction to occur, the interacting waves must be *locally* resonant. It may be necessary to include nonvariational effects [14] in adiabatic (e.g., amplitude equation) descriptions of mode interactions to incorporate this temporal locality. The temporal locality of the interactions may stem from the high dissipation in this system, as

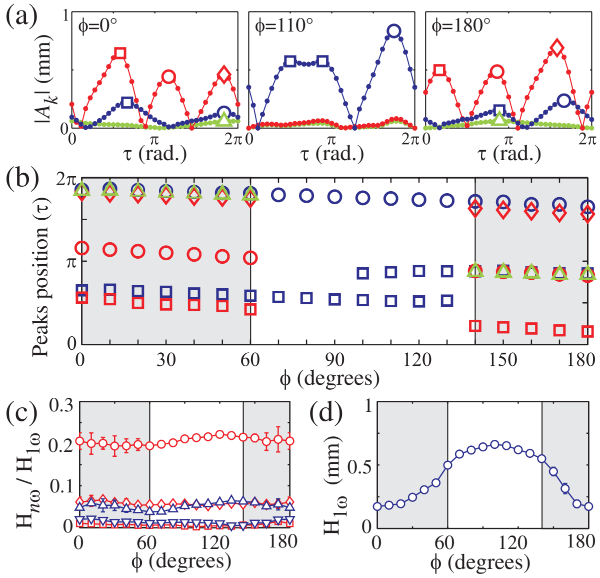

FIG. 4 (color online). Interactions between modes only occur when local peaks are aligned. Driving amplitudes are fixed at point iv within the 2MS region in Fig. 1(a), while ϕ is varied. Shaded (white) regions in (b)–(d) indicate 2MS (2ω-Squares). (a) Representative plots of $|A_k(\tau)|$ for different values of ϕ. (b) The temporal locations (in τ), of each local peak of $A_{k_{2\omega}}(\tau)$, $A_{k_{3\omega}}(\tau)$, and $A_{\mathrm{dif}}(t)$ as a function of ϕ. Symbols denoting the different values of k correspond to those in (a). The 2MS state occurs when local peaks of $|A_{k_{2\omega}}(\tau)|$ and $|A_{k_{3\omega}}(\tau)|$ are aligned for $0° \leq \phi \leq 60°$ (left) and again for $140° \leq \phi \leq 180°$, where alignment occurs for a different τ (right). When there is no alignment ($60° < \phi < 140°$) (middle), 2ω-Squares are observed. As in Fig. 2, all temporal harmonics are slaved to the fundamental harmonic, $H_{1\omega}$, despite large variations in its amplitude. (c) Amplitudes of odd (dark/blue line) and even (light/red line) harmonics as a function of ϕ, when scaled by $H_{1\omega}$. Symbols are as in Fig. 2(c). (d) $H_{1\omega}$ as a function of ϕ.

waves in this system are damped within a single excitation period (Q-value <1). Such highly damped systems do not possess the extended "memory" of systems whose Q-values are high, and temporal correlations are necessarily short. These results may be related to the role of the phase on the selection of patterns [15], and should be generally relevant for highly damped nonlinear systems.

An additional, rather surprising, result of this work is the harmonic composition of the nonlinear waves that are excited. Each of the different excited states in this system is composed of highly nonlinear waves (of amplitude A_k), which undergo either self interactions to create square patterns or 3-wave interactions to form superlattice states. The fact that the harmonic composition of each A_k is, in every case, functionally described by the Hill's equation is highly nontrivial, as this equation is strictly valid *only* in the perturbative limit (i.e., linear waves) for nondissipative systems. Furthermore, this precise harmonic structure and resulting dependence on the driving parameters are wholly independent of the nonlinear interactions that are taking place. This result may be generally relevant to parametrically driven nonlinear systems.

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