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SU(3) partial dynamical symmetry and nuclear shapes

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Abstract. We consider several variants of SU(3) partial dynamical symmetry in relation to quadrupole shapes in nuclei. Explicit construction of Hamiltonians with such property is presented in the framework of the interacting boson model (IBM), including higher order terms, and in its proton–neutron extension (IBM-2). The cases considered include a single prolate-deformed shape with solvable ground and γ or β bands, coexisting prolate-oblate shapes with solvable ground bands, and aligned axially-deformed proton–neutron shapes with solvable symmetric ground and γ bands and mixed-symmetry scissors and γ bands.

1 Introduction

Symmetries play a key role in nuclei by providing quantum numbers for the classification of states, determining spectral degeneracies and selection rules, and facilitating the calculation of matrix elements. Models based on spectrum generating algebras form a convenient framework to study their impact and have been used extensively in nuclear spectroscopy. Notable examples include Elliott's SU(3) model [1,2], symplectic model [3], pseudo SU(3) model [4], monopole and quadrupole pairing models [5], interacting boson models for even-even nuclei [6] and boson-fermion models for odd-mass nuclei [7]. In such models, the Hamiltonian is expanded in elements of a Lie algebra (G_{dyn}), called the dynamical (spectrum generating) algebra, in terms of which any operator of a physical observable can be expressed. A dynamical symmetry (DS) occurs if the Hamiltonian can be written in terms of the Casimir operators of a chain of nested algebras [8], $G_{dyn} \supset G_1 \supset G_2 \supset \ldots \supset G_{sym}$, terminating in the symmetry algebra G_{sym}. In such a case, the spectrum is completely solvable and the eigenstates, $|\lambda_{dyn}, \lambda_1, \lambda_2, \ldots, \lambda_{sym}\rangle$, are labeled by quantum numbers which are the labels of irreducible representations (irreps) of the algebras in the chain. A given G_{dyn} can encompass several DS chains, each providing characteristic analytic expressions for observables and definite selection rules for transition processes. An attractive feature of such models is that they are amenable to both quantum and classical treatments. The classical limit is obtained by introducing coherent (or intrinsic) states [9,10], which form a basis for studying the geometry of algebraic models and their relation to intuitive notions of shapes and excitation modes.

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A comprehensive framework for exploring the interplay of shapes and symmetries in nuclei, is provided by the interacting boson model (IBM) [6]. The latter describes low-lying quadrupole collective states in nuclei in terms of N monopole (s) and quadrupole (d) bosons, representing valence nucleon pairs. The model is based on a spectrum generating algebra $G_{\text{dyn}} = \text{U}(6)$ and a symmetry algebra $G_{\text{sym}} = \text{SO}(3)$. The Hamiltonian is expanded in the elements of U(6), $\{s^\dagger s, s^\dagger d_m, d_m^\dagger s, d_m^\dagger d_{m'}\}$, and consists of Hermitian, rotational-scalar interactions which conserve the total number of s- and d- bosons, $\hat{N} = \hat{n}_s + \hat{n}_d = s^\dagger s + \sum_m d_m^\dagger d_m$. The solvable limits of the IBM correspond to the following DS chains,

$$U(6) \supset U(5) \supset SO(5) \supset SO(3) \qquad |[N], n_d, \tau, n_\Delta, L\rangle, \qquad (1a)$$

$$U(6) \supset SU(3) \supset SO(3) \qquad |[N], (\lambda, \mu), K, L\rangle, \qquad (1b)$$

$$U(6) \supset \overline{SU(3)} \supset SO(3) \qquad |[N], (\bar{\lambda}, \bar{\mu}), \bar{K}, L\rangle, \qquad (1c)$$

$$U(6) \supset SO(6) \supset SO(5) \supset SO(3) \qquad |[N], \sigma, \tau, n_\Delta, L\rangle. \qquad (1d)$$

Here $N, n_d, (\lambda, \mu), (\bar{\lambda}, \bar{\mu}), \sigma, \tau, L$, label the relevant irreps of U(6), U(5), SU(3), $\overline{SU(3)}$, SO(6), SO(5), SO(3), respectively, and $K, \bar{K}, n_\Delta$, are multiplicity labels. The resulting spectra of these DS chains with leading sub-algebras G_1: U(5), SU(3), $\overline{SU(3)}$ and SO(6), resemble known paradigms of nuclear collective structure: spherical vibrator, prolate-, oblate- and γ-soft deformed rotors, respectively. Electromagnetic moments and rates can be calculated with transition operators of appropriate rank. For example, the one-body $E2$ operator reads

$$\hat{T}(E2) = e_{\text{b}}[d^\dagger s + s^\dagger \tilde{d} + \chi (d^\dagger \tilde{d})^{(2)}], \qquad (2)$$

where $\tilde{d}_m = (-1)^m d_{-m}$, and standard notation of angular momentum coupling is used.

A geometric visualization of the IBM is obtained by an energy surface,

$$E_N(\beta, \gamma) = \langle \beta, \gamma; N|\hat{H}|\beta, \gamma; N\rangle, \qquad (3)$$

defined by the expectation value of the Hamiltonian in a coherent (intrinsic) state of the form [11,12],

$$|\beta, \gamma; N\rangle = (N!)^{-1/2}[b_c^\dagger(\beta, \gamma)]^N |0\rangle, \qquad (4a)$$

$$b_c^\dagger(\beta, \gamma) = (1 + \beta^2)^{-1/2}[\beta \cos\gamma d_0^\dagger + \beta \sin\gamma (d_2^\dagger + d_{-2}^\dagger)/\sqrt{2} + s^\dagger]. \qquad (4b)$$

Here (β, γ) are quadrupole shape parameters whose values, $(\beta_{\text{eq}}, \gamma_{\text{eq}})$, at the global minimum of $E_N(\beta, \gamma)$ define the equilibrium shape for a given Hamiltonian. The shape can be spherical ($\beta = 0$) or deformed ($\beta > 0$) with $\gamma = 0$ (prolate), $\gamma = \pi/3$ (oblate), $0 < \gamma < \pi/3$ (triaxial) or γ-independent. The equilibrium deformations associated with the DS limits conform with their geometric interpretation and are given by,

$$U(5): \qquad \beta_{\text{eq}} = 0 \qquad\qquad n_d = 0, \qquad (5a)$$

$$SU(3): \qquad (\beta_{\text{eq}} = \sqrt{2}, \gamma_{\text{eq}} = 0) \qquad (\lambda, \mu) = (2N, 0), \qquad (5b)$$

$$\overline{SU(3)}: \qquad (\beta_{\text{eq}} = \sqrt{2}, \gamma_{\text{eq}} = \pi/3) \qquad (\bar{\lambda}, \bar{\mu}) = (0, 2N), \qquad (5c)$$

$$SO(6): \qquad (\beta_{\text{eq}} = 1, \gamma_{\text{eq}} \text{ arbitrary}) \qquad \sigma = N. \qquad (5d)$$

The coherent state $|\beta_{eq}, \gamma_{eq}; N\rangle$ (4), with the equilibrium deformations, serves as the intrinsic state for the ground band and is a lowest (or highest) weight state in a particular irrep of the leading sub-algebra in each of the chains (1). The DS Hamiltonians support a single minimum in their energy surface, hence serve as benchmarks for the dynamics of a single quadrupole shape.

An exact dynamical symmetry (DS) provides considerable insights into complex dynamics and its merits are self-evident. However, in most applications to realistic systems, its predictions are rarely fulfilled and one is compelled to break it. More often one finds that the assumed symmetry is not obeyed uniformly, i.e., is fulfilled by some of the states but not by others. The need to address such situations, but still preserve important symmetry remnants, has motivated the introduction of partial dynamical symmetry (PDS) [13]. The essential idea is to relax the stringent conditions of *complete* solvability so that only part of the eigenspectrum retains analyticity and/or good quantum numbers. In the present contribution, we present an explicit construction of Hamiltonians with the PDS property in the framework of the IBM, with applications to nuclear spectroscopy. We focus the discussion on PDS associated with SU(3) symmetry which, since the pioneering work by Elliott [1,2], has become an essential ingredient in all shell-model inspired models of rotational motion in nuclei and a central theme in nuclear physics [14].

2 Partial dynamical symmetry for a single shape

An algorithm for constructing Hamiltonians with a single PDS has been developed in [15] and further elaborated in [16]. In the IBM, the analysis starts from a dynamical symmetry chain,

$$\text{U}(6) \supset \text{G}_1 \supset \text{G}_2 \supset \ldots \supset \text{SO}(3) \qquad |[N], \lambda_1, \lambda_2, \ldots, L\rangle \qquad (\beta_{eq}, \gamma_{eq}), \qquad (6)$$

with leading sub-algebra G_1, related basis $|[N], \lambda_1, \lambda_2, \ldots, L\rangle$ and associated shape $(\beta_{eq}, \gamma_{eq})$. A number-conserving Hamiltonian with G_1 partial symmetry is found by writing it as an expansion in terms of G_1-tensors

$$\hat{H} = \sum_{\alpha, \beta} u_{\alpha\beta} \hat{T}_\alpha^\dagger \hat{T}_\beta, \qquad (7)$$

which annihilate the states in a particular G_1-irrep, $\lambda_1 = \Lambda_0$,

$$\hat{T}_\alpha |[N], \lambda_1 = \Lambda_0, \lambda_2, \ldots, L\rangle = 0. \qquad (8)$$

Equivalently, $\hat{T}_\alpha$ annihilate the lowest or highest weight state of this irrep,

$$\hat{T}_\alpha |[N], \lambda_1 = \Lambda_0\rangle = 0, \qquad (9)$$

from which the different L-states are obtained by projection. If $\lambda_1 = \Lambda_0$ is the ground-state irrep, the extremal state of equation (9) coincides with the intrinsic state for the ground band, $|\beta_{eq}, \gamma_{eq}; N\rangle$ (4), representing the equilibrium shape $(\beta_{eq}, \gamma_{eq})$. The G_1-tensors in equation (7), involve n-boson creation and annihilation operators with definite character under the chain (6). If both $\hat{T}_\alpha$ and the states $|[N], \lambda_1 = \Lambda_0, \lambda_2, \ldots, L\rangle$ span entire irreps of G_1, then the condition (8) is satisfied if their Kronecker product does not contain G_1 irreps which belong to the $[N-n]$ irrep of U(6).

In general, the Hamiltonian $\hat{H}$ of equation (7) is not invariant under G_1, hence most of its eigenstates are mixed with respect to G_1. However, the normal-ordered form (7) and the conditions (8)-(9), ensure that it has a subset of solvable zero-energy eigenstates with good symmetry G_1, implying G_1 partial symmetry for $\hat{H}$. The degeneracy of these states is lifted, without affecting their wave functions, by adding to $\hat{H}$ the Hamiltonian,

$$\hat{H}_c = \sum_{G_i \subset G_1} a_{G_i} \hat{C}[G_i], \tag{10}$$

composed of the Casimir operators of sub-algebras of G_1 in the chain (6). The states $|[N], \lambda_1 = \Lambda_0, \lambda_2, \ldots, L\rangle$ remain solvable eigenstates of the complete Hamiltonian,

$$\hat{H}_{\text{PDS}} = \hat{H} + \hat{H}_c = \hat{H}_{\text{DS}} + \hat{V}_0, \tag{11}$$

which, by definition, has G_1-PDS. The decomposition of equation (11) is referred to as a resolution of the Hamiltonian into intrinsic ($\hat{H}$) and collective ($\hat{H}_c$) parts [17,18]. The former determines the energy surface (3) and band-structure, while the latter determines the in-band rotational splitting. For specific choice of parameters, $u_{\alpha\beta}$ in equation (7), $\hat{H}$ reduces to the Casimir operators of G_1, which when combined with $\hat{H}_c$ comprise the DS Hamiltonian ($\hat{H}_{\text{DS}}$). The PDS Hamiltonian ($\hat{H}_{\text{PDS}}$) can then be written as $\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \hat{V}_0$, where V_0 contains the remaining terms in $\hat{H}$. In what follows, we review the SU(3)-DS limit of the IBM, construct Hamiltonians with SU(3)-PDS and show their relevance to nuclear spectroscopy.

3 SU(3) dynamical symmetry including higher-order terms

The SU(3)-DS limit is appropriate to the dynamics of a prolate-deformed shape. Its related chain, quantum numbers and equilibrium deformations are given by

$$\text{U(6)} \supset \text{SU(3)} \supset \text{SO(3)} \qquad |[N], (\lambda, \mu), K, L\rangle \qquad (\beta_{\text{eq}} = \sqrt{2}, \gamma_{\text{eq}} = 0). \tag{12}$$

For a given N, the allowed SU(3) irreps are $(\lambda, \mu) = (2N - 4k - 6m, 2k)$ with k, m, non-negative integers. The values of L contained in a given (λ, μ)-irrep follow the SU(3) $\supset$ SO(3) reduction rules [6] and the multiplicity label K corresponds geometrically to the projection of the angular momentum on the symmetry axis. The states $|[N], (\lambda, \mu), K, L\rangle$ form the (non-orthogonal) Elliott basis and the Vergados basis $|[N], (\lambda, \mu), \tilde{\chi}, L\rangle$ is obtained from it by a standard orthogonalization procedure. The two bases coincide in the large-N limit. The generators of SU(3) are the angular momentum operators, $L^{(1)}$, and the quadrupole operators, $Q^{(2)}$,

$$Q_m^{(2)} = d_m^\dagger s + s^\dagger \tilde{d}_m - \tfrac{1}{2}\sqrt{7}(d^\dagger \tilde{d})_m^{(2)}, \quad L_m^{(1)} = \sqrt{10}\,(d^\dagger \tilde{d})_m^{(1)}. \tag{13}$$

The quadratic and cubic Casimir operators of SU(3) are given by

$$\hat{C}_2[\text{SU(3)}] = 2Q^{(2)} \cdot Q^{(2)} + \tfrac{3}{4}L^{(1)} \cdot L^{(1)} \tag{14a}$$

$$\hat{C}_3[\text{SU(3)}] = -4\sqrt{7}Q^{(2)} \cdot (Q^{(2)} \times Q^{(2)})^{(2)} - \tfrac{9}{2}\sqrt{3}Q^{(2)} \cdot (L^{(1)} \times L^{(1)})^{(2)}, \tag{14b}$$

where the centered dot implies a scalar product and $\hat{C}_k[\mathrm{G}]$ denotes the Casimir operator of G of order k. The respective eigenvalues are

$$f_2(\lambda, \mu) = \lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu, \tag{15a}$$

$$f_3(\lambda, \mu) = (\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3). \tag{15b}$$

The SU(3) DS Hamiltonian involves a linear combination of $\hat{C}_2[\mathrm{SU}(3)]$, $\hat{C}_3[\mathrm{SU}(3)]$ and $\hat{C}_2[SO(3)] = L^{(1)} \cdot L^{(1)}$. The spectrum is completely solvable and resembles that of an axially-deformed roto-vibrator composed of SU(3) (λ, μ)-multiplets forming rotational bands with $L(L+1)$-splitting. The lowest irrep $(2N, 0)$ contains the ground band $g(K = 0)$ of a prolate deformed nucleus. The first excited irrep $(2N-4, 2)$ contains both the $\beta(K = 0)$ and $\gamma(K = 2)$ bands. States with the same L in different K-bands are degenerate.

Three-body terms allow an additional solvable symmetry-conserving operator,

$$\hat{\Omega} = -4\sqrt{3}\, Q^{(2)} \cdot (L^{(1)} \times L^{(1)})^{(2)}. \tag{16}$$

This operator is constructed from the SU(3) generators, hence is diagonal in (λ, μ). It breaks, however, the aforementioned K-degeneracy. A well-defined procedure exists for obtaining the eigenstates of $\hat{\Omega}$ and corresponding eigenvalues $\langle\hat{\Omega}\rangle$ [19,20]. In particular, for the irreps $(\lambda, 0)$ and $(\lambda, 2)$ with λ even, we have

$(\lambda, 0)\ K = 0,\ L = 0, 2, 4, \ldots, \lambda:$ $\quad\langle\hat{\Omega}\rangle = (2\lambda + 3)L(L+1),$ (17a)

$(\lambda, 2)\ K = 2,\ L = 3, 5, 7, \ldots, \lambda+1, \lambda+2:$ $\langle\hat{\Omega}\rangle = (2\lambda + 5)[L(L+1) - 12],$ (17b)

$(\lambda, 2)\ K = 0,\ L = 0:$ $\quad\langle\hat{\Omega}\rangle = 0,$ (17c)

$(\lambda, 2)\ K = 0, 2,\ \ L = 2, 4, 6, \ldots, \lambda:$

$$\langle\hat{\Omega}\rangle = (2\lambda + 5)[L(L+1) - 6] \pm 6\sqrt{(2\lambda + 5)^2 + L(L+1)(L-1)(L+2)}. \tag{17d}$$

The two eigenstates $|(\lambda, \mu), L, \pm\rangle$ corresponding to the eigenvalues in equation (17d), involve a mixture of $K = 0, 2$, Elliott states with the β band lying above the γ band. Several works have examined the influence of the operator $\hat{\Omega}$ (16) on nuclear spectra, within the IBM [20–22] and the symplectic shell model [23,24].

4 SU(3) partial dynamical symmetry for a single shape

According to Section 2, the method to construct Hamiltonians with SU(3)-PDS is based on the identification of SU(3) tensors which annihilate states in a given SU(3) irrep (λ, μ), chosen here to be the ground band irrep $(2N, 0)$. The tensors involve n-boson operators with definite character under the SU(3) chain (12),

$$\hat{B}^{\dagger}_{[n](\lambda,\mu)\tilde{\chi}\ell m}, \quad \tilde{B}_{[n^5](\mu,\lambda)\tilde{\chi}\ell m} \equiv (-)^m (\hat{B}^{\dagger}_{[n](\lambda,\mu)\tilde{\chi}\ell,-m})^{\dagger}. \tag{18}$$

The SU(3) tensor operators for $n = 2$, span the irrep $(\lambda, \mu) = (0, 2)$ and are given by,

$$B^{\dagger}_{[2](0,2)0;00} = \tfrac{1}{3\sqrt{2}} P^{\dagger}_0, \quad B^{\dagger}_{[2](0,2)0;2m} = \tfrac{1}{3\sqrt{2}} P^{\dagger}_{2m}. \tag{19}$$

They are constructed of boson-pair operators with angular momentum $\ell = 0, 2$,

$$P_0^\dagger = d^\dagger \cdot d^\dagger - 2(s^\dagger)^2, \tag{20a}$$

$$P_{2m}^\dagger = 2d_m^\dagger s^\dagger + \sqrt{7}\,(d^\dagger d^\dagger)_m^{(2)}. \tag{20b}$$

The SU(3) tensor operators for $n = 3$, span the irreps $(\lambda, \mu) = (2, 2)$ and $(0, 0)$,

$$\hat{B}_{[3](2,2)0;00}^\dagger = \tfrac{1}{30}W_0^\dagger, \quad \hat{B}_{[3](2,2)2;2m}^\dagger = \tfrac{1}{\sqrt{78}}W_{2m}^\dagger, \quad \hat{B}_{[3](2,2)0;2m}^\dagger = \tfrac{1}{\sqrt{910}}V_{2m}^\dagger,$$

$$\hat{B}_{[3](2,2)2;3m}^\dagger = \tfrac{1}{\sqrt{30}}W_{3m}^\dagger, \quad \hat{B}_{[3](2,2)2;4m}^\dagger = \tfrac{1}{\sqrt{30}}W_{4m}^\dagger, \quad \hat{B}_{[3](0,0)0;00}^\dagger = \tfrac{1}{18}\Lambda^\dagger, \tag{21}$$

and are obtained by coupling $s^\dagger$ and $d_m^\dagger$ to the $n = 2$ SU(3) tensors of equation (20),

$$W_0^\dagger = 5P_0^\dagger s^\dagger - P_2^\dagger \cdot d^\dagger, \quad W_{2m}^\dagger = P_0^\dagger d_m^\dagger + 2P_{2m}^\dagger s^\dagger, \quad V_{2m}^\dagger = 6P_0^\dagger d_m^\dagger - P_{2m}^\dagger s^\dagger,$$

$$W_{3m}^\dagger = (P_2^\dagger d^\dagger)_m^{(3)}, \quad W_{4m}^\dagger = (P_2^\dagger d^\dagger)_m^{(4)}, \quad \Lambda^\dagger = P_0^\dagger s^\dagger + P_2^\dagger \cdot d^\dagger. \tag{22}$$

It should be noted that $P_0^\dagger d_m^\dagger + P_{2m}^\dagger s^\dagger = -\tfrac{\sqrt{7}}{2}(P_2^\dagger d^\dagger)_m^{(2)}$ and $W_{3m}^\dagger = \sqrt{7}(d^\dagger d^\dagger)^{(2)}d^\dagger)_m^{(3)}$.

The Hermitian conjugate of the operators in equation (20), P_0 and P_{2m}, transform as $(2, 0)$ under SU(3) and satisfy

$$P_0 \,|[N], (2N, 0), K = 0, L\rangle = 0, \tag{23a}$$

$$P_{2m} \,|[N], (2N, 0), K = 0, L\rangle = 0, \qquad L = 0, 2, 4, \ldots, 2N. \tag{23b}$$

These relations follow from the fact that the action of the operators $P_{\ell m}$ leads to a state with $N - 2$ bosons in the U(6) irrep $[N - 2]$, which does not contain the SU(3) irreps obtained from the product $(2, 0) \times (2N, 0)$. The indicated L-states in equation (23) span the entire SU(3) irrep $(\lambda, \mu) = (2N, 0)$ and form the rotational members of the ground band. They can be obtained by SO(3) projection from the intrinsic state for the SU(3) ground band,

$$|g; (2N, 0)K = 0; N\rangle = |\beta_{\text{eq}} = \sqrt{2}, \gamma_{\text{eq}} = 0; N\rangle, \tag{24}$$

defined as in equation (4) with the SU(3) equilibrium deformations and representing a prolate-deformed shape.

The relations of equation (23) ensure that the Hermitian conjugate of all SU(3) tensors in equations (19) and (21), P_0, P_{2m}, $W_{\ell m}$, V_{2m}, Λ, annihilate $|g; (2N, 0)K = 0; N\rangle$ and the corresponding L-projected states. By the procedure of equation (7), the most general (2+3)-body Hamiltonian with SU(3) partial symmetry can be written as,

$$\hat{H} = h_2\,P_2^\dagger \cdot \tilde{P}_2 + h_0\,P_0^\dagger P_0 + t_0^a\,\Lambda^\dagger \Lambda + t_0^b\,W_0^\dagger W_0 + t_0^c\,(\Lambda^\dagger W_0 + W_0^\dagger \Lambda)$$
$$+ t_2^a\,V_2^\dagger \cdot \tilde{V}_2 + t_2^b\,W_2^\dagger \cdot \tilde{W}_2 + t_2^c\,(V_2^\dagger \cdot \tilde{W}_2 + W_2^\dagger \cdot \tilde{V}_2) + t_3\,W_3^\dagger \cdot \tilde{W}_3 + t_4\,W_4^\dagger \cdot \tilde{W}_4, \tag{25}$$

and has a subset of SU(3) basis states, $|[N], (2N, 0), K = 0, L\rangle$, as zero-energy eigenstates. Several SU(3)-preserving interactions are contained in the expression (25). Specifically, the (integrity basis) term,

$$\hat{\Omega} - (4\hat{N} + 3)L^{(1)} \cdot L^{(1)} = 2(\hat{N} - 2)P_0^\dagger P_0 - 2(2\hat{N} - 1)P_2^\dagger \cdot \tilde{P}_2 + 2\Lambda^\dagger \Lambda$$
$$- 2(\Lambda^\dagger s P_0 + P_0^\dagger s^\dagger \Lambda) + 2\,W_2^\dagger \cdot \tilde{W}_2 + 4\,W_3^\dagger \cdot \tilde{W}_3, \tag{26}$$

and the following SU(3)-scalar terms,

$$\hat{\theta}_2 \equiv -\hat{C}_2[\text{SU(3)}] + 2\hat{N}(2\hat{N} + 3) = P_0^\dagger P_0 + P_2^\dagger \cdot \tilde{P}_2, \tag{27a}$$

$$(\hat{N} - 2)\hat{\theta}_2 = 6 \left[5 \sum_{[\tilde{\chi};L]} B_{[3](2,2)\tilde{\chi};L}^\dagger \cdot \tilde{B}_{[3^5](2,2)\tilde{\chi};L} + 9 B_{[3](0,0)0;0}^\dagger \tilde{B}_{[3^5](0,0)0;0} \right], \tag{27b}$$

$$\hat{C}_3[\text{SU(3)}] + (2\hat{N} + 3) \left[3\hat{\theta}_2 - 2\hat{N}(4\hat{N} + 3) \right] = 2\Lambda^\dagger \Lambda. \tag{27c}$$

The three terms in equation (27) are related to the Casimir operators of SU(3), equation (14), hence are naturally included in the dynamical symmetry Hamiltonian. Allowing N-dependent coefficients, the latter can be transcribed in the form,

$$\hat{H}_{\text{DS}} = h_1 \Lambda^\dagger \Lambda + h_2 \hat{\theta}_2 + C \hat{C}_2[\text{SO(3)}]. \tag{28a}$$
$$E_{\text{DS}} = [h_2 + \tfrac{3}{2} h_1 (2N + 3)] [-f_2(\lambda, \mu) + 2N(2N + 3)]$$
$$+ \tfrac{1}{2} h_1 [f_3(\lambda, \mu) - 2N(2N + 3)(4N + 3)] + C L(L + 1). \tag{28b}$$

The SU(3)-PDS Hamiltonian can then be written as in equation (11),

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + h_3 P_0^\dagger P_0 + h_4 P_0^\dagger s^\dagger s P_0 + h_5 \left(\Lambda^\dagger s P_0 + P_0^\dagger s^\dagger \Lambda \right)$$
$$+ h_6 W_2^\dagger \cdot \tilde{W}_2 + h_7 W_3^\dagger \cdot \tilde{W}_3 + h_8 s^\dagger P_2^\dagger \cdot \tilde{P}_2 s. \tag{29}$$

In general, $\hat{H}_{\text{PDS}}$ (29) does not preserve SU(3) yet, by construction, for any choice of parameters, it has a solvable ground band with good SU(3) symmetry,

$$g : |[N], (2N, 0), K = 0, L\rangle \quad L = 0, 2, 4, \ldots, 2N \tag{30a}$$
$$E_{\text{PDS}} = CL(L + 1). \tag{30b}$$

For specific choices, additional solvable states are obtained. One class of SU(3)-PDS Hamiltonians is obtained by choosing in equation (29) the h_3, h_4, h_5 terms, involving the operators $P_0^\dagger$ (20) and $\Lambda^\dagger$ (22). P_0 satisfies equation (23a) and in addition,

$$P_0 |[N], (\lambda, \mu) = (2N - 4k, 2k), K = 2k, L\rangle = 0, \quad L = K, K + 1, \ldots, (2N - 2k). \tag{31}$$

For $k > 0$, the indicated L-states span only part of the SU(3) irreps $(\lambda, \mu) = (2N - 4k, 2k)$ and form the rotational members of excited $\gamma^k (K = 2k)$ bands. This result follows from the fact that P_0 annihilates the intrinsic states of these bands,

$$|\gamma_{K=2k}^k; (2N - 4k, 2k)K = 2k); N\rangle = \sqrt{\tfrac{3^k(2N-4k+1)!!}{k!(2N-2k+1)!!}} (B_{\gamma,2}^\dagger)^k |g; N - 2k\rangle, \tag{32}$$

where $B_{\gamma,2}^\dagger = \frac{1}{3\sqrt{2}} P_{2,2}^\dagger$ and $|g; N - 2k\rangle$ is obtained from equation (24). The operator Λ transforms as $(0,0)$ under SU(3) and annihilates *all* states in the irreps $(2N - 4k, 2k)$,

$$\Lambda |[N], (2N - 4k, 2k), K, L\rangle = 0. \tag{33}$$

This result follows from the fact that the U(6) irrep $[N - 3]$ does not contain SU(3) irreps obtained from the product $(0,0) \times (2N - 4k, 2k)$. The relations in equations (31)
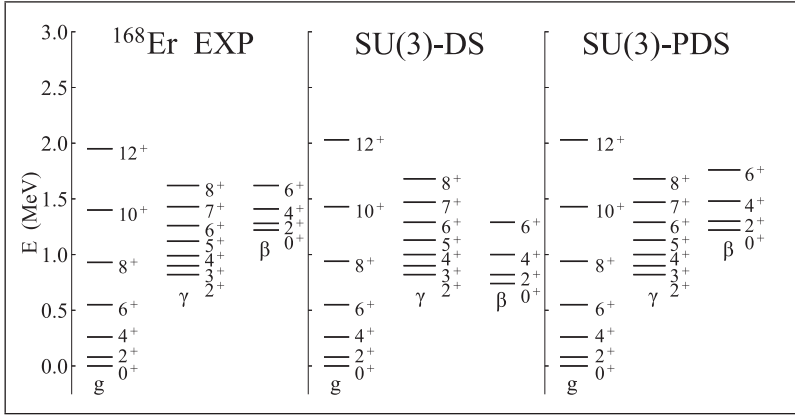
Fig. 1. Observed spectrum of ^{168}Er compared with SU(3)-DS and SU(3)-PDS calculations. The latter employ $\hat{H}_{DS}$ (28a) and $\hat{H}_{PDS}$ (34) with $h_2 = 4$, $C = 13$, $h_3 = 4$ keV (other $h_i = 0$) and $N = 16$. Adapted from [25].

and (33) ensure that the following Hamiltonian,

$$\hat{H}_{\text{PDS}} = \hat{H}_{DS} + h_3 P_0^\dagger P_0 + h_4 P_0^\dagger s^\dagger s P_0 + h_5 \left(\Lambda^\dagger s P_0 + P_0^\dagger s^\dagger \Lambda \right), \qquad (34)$$

has solvable ground band $g(K = 0)$, equation (30), and $\gamma^k(K = 2k)$ bands with good SU(3) symmetry,

$$(\gamma^k)_{K=2k}: \ |[N], (2N - 4k, 2k), K = 2k, L\rangle \ \ L = K, K+1, K+2, \ldots, (2N-2k), \quad (35a)$$
$$E_{PDS} = h_2 \, 6k(2N - 2k + 1) + CL(L + 1). \qquad (35b)$$

The remaining eigenstates of $\hat{H}_{PDS}$, in particular, members of the $\beta(K = 0)$ band, are mixed.

The experimental spectra of the ground $g(K = 0)$, $\gamma(K = 2)$ and $\beta(K = 0)$ bands in ^{168}Er is shown in Figure 1, and compared with an exact DS and PDS calculations [25], employing the Hamiltonian (34). The SU(3) PDS spectrum is clearly seen to be an improvement over the exact SU(3) DS description, since the β-γ degeneracy of the latter is lifted. The ground and gamma are still pure SU(3) bands, but the beta band is found to contain 13% admixtures into the dominant $(2N - 4, 2)$ irrep [26]. Since the wave functions of the solvable states are known, one has at hand *analytic* expressions for matrix elements of observables between them [27]. The $E2$ operator (2), can be transcribed as $\hat{T}(E2) = \alpha Q^{(2)} + \theta(d^\dagger s + s^\dagger \tilde{d})$, with $Q^{(2)}$ defined in equation (13). Since the solvable ground and gamma bands reside in different SU(3) irreps, $Q^{(2)}$ as a generator, cannot connect them and, consequently, $B(E2)$ ratios for $\gamma \to g$ transitions do no depend on the $E2$ parameters (α, θ) nor on parameters of the PDS Hamiltonian (34). Overall, as shown in Figure 2, these parameter-free predictions of SU(3)-PDS account well for the data in ^{168}Er [25,28]. Similar evidence for this type of SU(3)-PDS has been presented for other rare-earth and actinide nuclei [28–30], suggesting a wider applicability of this concept.

Another class of Hamiltonians with SU(3) PDS is obtained by choosing the h_6, h_7 terms in equation (29), involving the operators $W_{2m}^\dagger$ and $W_{3m}^\dagger$ (22). $W_{\ell m}$ ($\ell = 2, 3$) annihilate the states of the ground band, $|[N], (2N, 0), K = 0, L\rangle$, and in addition satisfy,
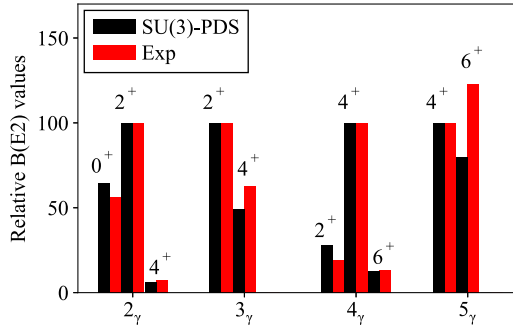
Fig. 2. Comparison of the SU(3)-PDS parameter-free predictions with the data on the relative $B(E2; L_\gamma \to L)$ values for $\gamma \to g$ $E2$ transitions in ^{168}Er. Adapted from [28].

$$W_{\ell m} \,|[N], (\lambda, \mu) = (2N - 4, 2), K = 0, L\rangle = 0, \quad L = 0, 2, 4, \dots, (2N - 4). \quad (36)$$

The indicated L-states span part of the irrep $(2N - 4, 2)$ of SU(3) and comprise the $\beta(K = 0)$ band. This result follows from the fact that $W_{\ell m}$ annihilate the intrinsic state of this band,

$$|\beta; N\rangle = \sqrt{\tfrac{3}{2N-1}} \, B_\beta^\dagger |g; N - 2\rangle \ , \quad B_\beta^\dagger = \tfrac{1}{3\sqrt{6}} (\sqrt{2} P_0^\dagger - P_{2,0}^\dagger), \quad (37)$$

where $|g; N - 2\rangle$ is obtained from equation (24). This property ensures that the following Hamiltonian [31],

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + h_6 \, W_2^\dagger \cdot \tilde{W}_2 + h_7 \, W_3^\dagger \cdot \tilde{W}_3, \quad (38)$$

has a solvable ground band $g(K = 0)$, equation (30), and a solvable $\beta(K = 0)$ band,

$$\beta : \ |[N], (2N - 4, 2), K = 0, L\rangle \quad L = 0, 2, 4, \dots, 2N \quad (39a)$$

$$E_{PDS} = h_2 \, 6(2N - 1) + CL(L + 1). \quad (39b)$$

Other bands, in particular the γ band, are mixed. For the special case, $h_7 = 2h_6$, $\hat{H}_{\text{PDS}}$ (38) has additional solvable states, $|[N], (2N - 4, 2), K = 2, L\rangle$ with $L = 3, 5, 7, \dots, (2N - 3), 2N - 2$, and energy $h_2 \, 6(2N - 1) + h_6 \, 3[8(N - 1)^2 - L(L + 1)]$, a result obtained by combining equations (17b) and (26).

A comparison, in Figure 3, of the experimental spectrum of ^{156}Gd with an SU(3)-DS calculation, shows a good description for properties of states in the ground and β bands, however, the resulting fit to energies of the γ-band is quite poor. The latter are not degenerate with the β band and, moreover, display an odd-even staggering with pronounced deviations from a rigid-rotor $L(L+1)$ pattern. This effect can be visualized by plotting the quantity $Y(L)$, defined in the caption of Figure 4. For a rotor this quantity is flat, $Y(L) = 0$, as illustrated with the SU(3) DS calculation, which is in marked disagreement with the empirical data. In the PDS calculation, the gamma band contains 15% SU(3) admixtures into the dominant $(2N - 4, 2)$ irrep and the empirical odd–even staggering is well reproduced [31]. The PDS results for the γ band are obtained without affecting the solvability and SU(3) purity of states in the ground and beta bands. Since the wave functions of the latter states are known, one has at hand analytic expressions for $E2$ transitions between them [27], that can be
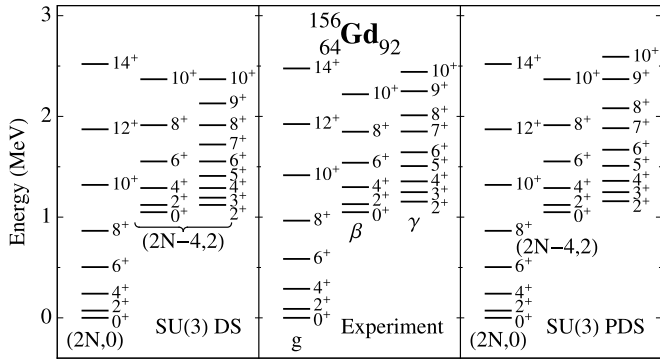
Fig. 3. Observed spectrum of ^{156}Gd compared with SU(3)-DS and SU(3)-PDS calculations. The latter employ $\hat{H}_{DS}$ (28a) and $\hat{H}_{PDS}$ (38) with $h_1 = 0$, $h_2 = 7.6$, $C = 12.0$, $h_6 = -0.23$, $h_7 = 1.54$ keV and $N = 12$. Adapted from [31].
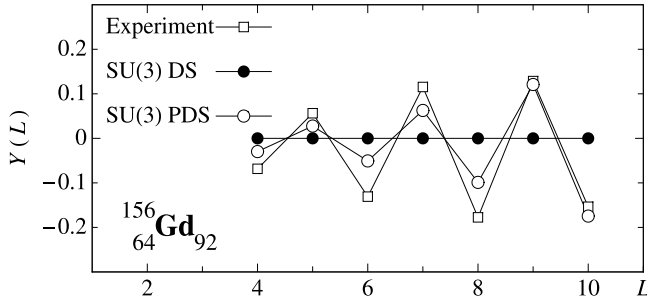

Fig. 4. Observed and calculated (SU(3)-DS and SU(3)-PDS) odd-even staggering of the γ band in ^{156}Gd. Here $Y(L) = \frac{2L-1}{L} \times \frac{E(L)-E(L-1)}{E(L)-E(L-2)} - 1$, where $E(L)$ is the energy of a γ-band level with angular momentum L. For SU(3)-DS, $Y(L) = 0$, as expected of a rigid rotor. Adapted from [31].

Table 1. Observed and calculated $B(E2)$ values in ^{156}Gd. For both the SU(3) DS and PDS calculations, the parameters of the E2 operator, equation (2), are $e_b = 0.166$ eb and $\chi = -0.168$ [31]. The indicated ranges in the experimental $B(E2)$ values for transitions from the β band, reflect uncertainties in the lifetime measurements [32].

L_i^π	L_f^π	Experiment	PDS	L_i^π	L_f^π	Experiment	PDS
2_1^+	0_1^+	0.933 25	0.933	0_β^+	2_1^+	$0.021 \to 0.055$	0.034
4_1^+	2_1^+	1.312 25	1.313	2_β^+	0_1^+	$0.0028 \to 0.0057$	0.0055
6_1^+	4_1^+	1.472 40	1.405	2_β^+	2_1^+	$0.016 \to 0.031$	0.0084
8_1^+	6_1^+	1.596 85	1.409	2_β^+	4_1^+	$0.018 \to 0.037$	0.020
10_1^+	8_1^+	1.566 70	1.364	4_β^+	2_1^+	$0.0047 \to 0.011$	0.0067
				4_β^+	4_1^+	$0.0098 \to 0.022$	0.0067
				4_β^+	6_1^+	$0.0080 \to 0.018$	0.021
				4_β^+	2_β^+	$1.00 \to 2.19$	0.951

used to test the SU(3)-PDS. As shown in Table 1, recent measurements of lifetimes for $E2$ decays from states of the β-band in ^{156}Gd [32], confirm the PDS predictions.
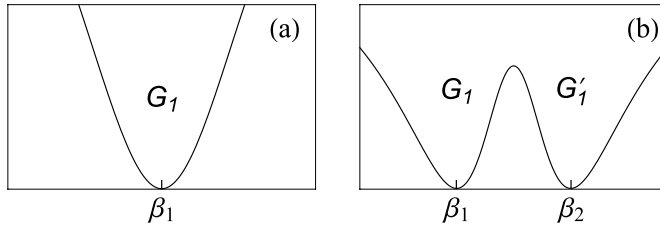
Fig. 5. Energy surfaces accommodating (a) a single minimum, associated with single G_1-DS or G_1-PDS; (b) double minima, associated with coexisting G_1-PDS and G_1'-PDS.

5 Multiple partial dynamical symmetries and shape coexistence

Coexistence of different shapes in the same nucleus is a ubiquitous phenomenon, known to occur widely across the nuclear chart [33]. It entails the presence in the spectrum of several states (or bands of states) at similar energies with distinct properties, reflecting the nature of their dissimilar dynamics. In the shell model description of nuclei near shell-closure, this phenomena is attributed to the occurrence of multi-particle multi-hole intruder excitations across shell gaps. In a mean-field based approach, the coexisting shapes are associated with different minima of an energy surface calculated self-consistently. The relevant Hamiltonians contain competing terms with different tendencies, resulting in a shape-phase transition in which the two minima cross and the underlying configurations interchange their roles.

From an algebraic perspective, the dynamics of a single shape is associated with a single dynamical symmetry, either exact or partial. The corresponding DS or PDS Hamiltonians support a single minimum in their energy surface at particular deformations. In a similar spirit, multiple shapes are associated with different dynamical symmetries (denoted by G_1, and G_1' in Fig. 5). The relevant Hamiltonians support multiple minima in their energy surface at particular deformations and mix these incompatible (non-commuting) symmetries. In such circumstances, exact DSs are broken, and any remaining symmetries can at most be partial, i.e., shared by only a subset of states. To explore such scenarios of persisting symmetries in relation to shape coexistence, requires an extension of the PDS algorithm of Section 2, to encompass a construction of Hamiltonians with several distinct PDSs [34–36]. We focus on the dynamics in the vicinity of the critical point, where the corresponding multiple minima in the energy surface are near-degenerate and the structure changes most rapidly.

For that purpose, consider two different shapes specified by equilibrium deformations (β_1, γ_1) and (β_2, γ_2) whose dynamics is described, respectively, by the following DS chains

$$\text{U(6)} \supset \text{G}_1 \supset \text{G}_2 \supset \ldots \supset \text{SO(3)} \qquad |[N], \lambda_1, \lambda_2, \ldots, L\rangle \qquad (\beta_1, \gamma_1), \quad (40\text{a})$$

$$\text{U(6)} \supset \text{G}_1' \supset \text{G}_2' \supset \ldots \supset \text{SO(3)} \qquad |[N], \sigma_1, \sigma_2, \ldots, L\rangle \qquad (\beta_2, \gamma_2), \quad (40\text{b})$$

with different leading sub-algebras $(G_1 \neq G_1')$ and associated bases. As portrayed in Figure 5, at the critical point, the corresponding minima representing the two shapes, and respective ground bands are degenerate. Accordingly, we consider an intrinsic critical-point Hamiltonian as in equation (7), $\hat{H} = \sum_{\alpha,\beta} u_{\alpha\beta} \hat{T}_\alpha^\dagger \hat{T}_\beta$, but now require the operators $\hat{T}_\alpha$ to satisfy simultaneously the following two conditions,

$$\hat{T}_\alpha |[N], \lambda_1 = \Lambda_0, \lambda_2, \ldots, L\rangle = 0, \tag{41a}$$

$$\hat{T}_\alpha |[N], \sigma_1 = \Sigma_0, \sigma_2, \ldots, L\rangle = 0. \tag{41b}$$

The states of equation (41a) reside in the $\lambda_1 = \Lambda_0$ irrep of G_1, are classified according to the DS-chain (40a), hence have good G_1 symmetry. Similarly, the states of equation (41b) reside in the $\sigma_1 = \Sigma_0$ irrep of G'_1, are classified according to the DS-chain (40b), hence have good G'_1 symmetry. Equivalently, $\hat{T}_\alpha$ annihilate the extremal states of both irreps,

$$\hat{T}_\alpha |[N], \lambda_1 = \Lambda_0\rangle = 0, \tag{42a}$$

$$\hat{T}_\alpha |[N], \sigma_1 = \Sigma_0\rangle = 0, \tag{42b}$$

from which the different L-states are obtained by projection. Although G_1 and G'_1 are incompatible, both sets of states in equations (41) are eigenstates of the same Hamiltonian. In general, $\hat{H}$ itself is not necessarily invariant under G_1 nor under G'_1 and, therefore, its other eigenstates can be mixed with respect to both G_1 and G'_1. If $\lambda_1 = \Lambda_0$ and $\sigma_1 = \Sigma_0$ are the ground-state irreps of G_1 and G'_1, the extremal states of equations (42) coincide with the intrinsic states $|\beta_1, \gamma_1; N\rangle$ and $|\beta_2, \gamma_2; N\rangle$, equation (4), of the two ground bands, representing the equilibrium shapes (β_1, γ_1) and (β_2, γ_2), respectively. Identifying, as in equation (10), the collective part of the Hamiltonian $(\hat{H}_c)$, with the Casimir operators of the algebras which are common to both chains (40), in particular $\hat{C}_2[\mathrm{SO}(3)]$, the two sets of states in equations (41) remain (non-degenerate) eigenstates of the complete critical-point Hamiltonian which has the form as in equation (11), $\hat{H}_{\mathrm{PDS}} = \hat{H} + \hat{H}_c = \hat{H}_{\mathrm{DS}} + \hat{V}_0$. The latter, by construction, has both G_1-PDS and G'_1-PDS. In what follows, we apply the above procedure to a case study of coexisting axially-deformed shapes with multiple PDSs.

6 SU(3) partial dynamical symmetry and prolate-oblate coexistence

The DS limits appropriate to prolate and oblate shapes correspond, respectively, to the chains [6],

$$\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3) \qquad |[N], (\lambda, \mu), K, L\rangle \qquad (\beta_{\mathrm{eq}} = \sqrt{2}, \gamma_{\mathrm{eq}} = 0), \tag{43a}$$

$$\mathrm{U}(6) \supset \overline{\mathrm{SU}(3)} \supset \mathrm{SO}(3) \qquad |[N], (\bar{\lambda}, \bar{\mu}), \bar{K}, L\rangle \qquad (\beta_{\mathrm{eq}} = \sqrt{2}, \gamma_{\mathrm{eq}} = \pi/3). \tag{43b}$$

The two chains have similar properties, but the classification of states is different. For a given N, the allowed $\overline{\mathrm{SU}(3)}$ irreps are $(\bar{\lambda}, \bar{\mu}) = (2k, 2N - 4k - 6m)$, with k, m, non-negative integers, compared to $(\lambda, \mu) = (2N - 4k - 6m, 2k)$ for SU(3). The $\overline{\mathrm{SU}(3)}$ algebra involves the following quadrupole and angular momentum operators,

$$\bar{Q}_m^{(2)} = d_m^\dagger s + s^\dagger \tilde{d}_m + \tfrac{1}{2}\sqrt{7}(d^\dagger \tilde{d})_m^{(2)}, \quad L^{(1)} = \sqrt{10}(d^\dagger \tilde{d})_m^{(1)}. \tag{44}$$

The generators of SU(3) and $\overline{\mathrm{SU}(3)}$, $Q^{(2)}$ (13) and $\bar{Q}^{(2)}$ (44), and corresponding basis states, $|[N], (\lambda, \mu), K, L\rangle$ and $|[N], (\bar{\lambda}, \bar{\mu}), \bar{K}, L\rangle$, are related by a change of phase

$(s^\dagger, s) \to (-s^\dagger, -s)$, induced by the operator

$$\mathcal{R}_s = \exp(i\pi\hat{n}_s), \quad \hat{n}_s = s^\dagger s. \tag{45}$$

The quadratic and cubic Casimir operators of $\overline{\mathrm{SU(3)}}$, $\hat{C}_2[\overline{\mathrm{SU(3)}}]$ and $\hat{C}_3[\overline{\mathrm{SU(3)}}]$, have the form as in equations (14) with $Q^{(2)}$ replaced by $\bar{Q}^{(2)}$ and eigenvalues $f_2(\bar{\lambda}, \bar{\mu})$ and $f_3(\bar{\lambda}, \bar{\mu})$, equations (15). As previously mentioned, in the SU(3)-DS, the prolate ground band $g(K = 0)$ has $(2N, 0)$ character and the $\beta(K = 0)$ and $\gamma(K = 2)$ bands have $(2N-4, 2)$. In the $\overline{\mathrm{SU(3)}}$-DS, the oblate ground band $g(\bar{K} = 0)$ has $(0, 2N)$ character and the excited $\beta(\bar{K} = 0)$ and $\gamma(\bar{K} = 2)$ bands have $(2, 2N-4)$. Henceforth, we denote such prolate and oblate bands by (g_1, β_1, γ_1) and (g_2, β_2, γ_2), respectively. Since $\mathcal{R}_s Q^{(2)} \mathcal{R}_s^{-1} = -\bar{Q}^{(2)}$, the SU(3) and $\overline{\mathrm{SU(3)}}$ DS spectra are identical but the quadrupole moments of corresponding states differ.

Following the procedure of Section 5, the intrinsic part of the critical-point Hamiltonian, relevant to prolate-oblate coexistence, is constructed of operators which annihilate both the SU(3) and $\overline{\mathrm{SU(3)}}$ ground bands. Specifically, $P_0^\dagger$ (20a), which is a $(0, 2)$ $[(2, 0)]$ tensor under SU(3) $[\overline{\mathrm{SU(3)}}]$ and $W_{3m}^\dagger$ (22), which is a $(2, 2)$ tensor under both algebras. These tensors commute with $\mathcal{R}_s$ (45) and satisfy,

$$P_0 \,|[N], (\lambda, \mu) = (2N, 0), K = 0, L\rangle = W_{3m} \,|[N], (\lambda, \mu) = (2N, 0), K = 0, L\rangle = 0, \tag{46a}$$

$$P_0 \,|[N], (\bar{\lambda}, \bar{\mu}) = (0, 2N), \bar{K} = 0, L\rangle = W_{3m} \,|[N], (\bar{\lambda}, \bar{\mu}) = (0, 2N), \bar{K} = 0, L\rangle = 0, \tag{46b}$$

with $L = 0, 2, 4, \dots 2N$. The states of equation (46a) comprise the SU(3) prolate ground band (g_1), projected from the intrinsic state, $|\beta_{\mathrm{eq}} = \sqrt{2}, \gamma_{\mathrm{eq}} = 0; N\rangle$, equation (4). Similarly, the states of equation (46b) comprise the $\overline{\mathrm{SU(3)}}$ oblate ground band (g_2), projected from the intrinsic state $|\beta_{\mathrm{eq}} = \sqrt{2}, \gamma_{\mathrm{eq}} = \pi/3; N\rangle$. The latter deformations are equivalent to $(\beta_{\mathrm{eq}} = -\sqrt{2}, \gamma_{\mathrm{eq}} = 0)$, since the intrinsic states $|\beta, \gamma = \pi/3; N\rangle$ and $|-\beta, \gamma = 0; N\rangle$ are related by a rotation by Euler angles $(\pi/2, \pi/2, \pi/2)$, hence contain the same L-states. The resulting intrinsic Hamiltonian is found to be [35],

$$\hat{H} = r_0 \, P_0^\dagger \hat{n}_s P_0 + r_2 \, P_0^\dagger \hat{n}_d P_0 + r_3 \, W_3^\dagger \cdot \tilde{W}_3. \tag{47}$$

The corresponding energy surface, $E_N(\beta, \gamma) = N(N-1)(N-2)\tilde{E}(\beta, \gamma)$, is given by

$$\tilde{E}(\beta, \gamma) = \left\{ (\beta^2 - 2)^2 \left[r_0 + r_2\beta^2 \right] + r_3\beta^6(1 - \Gamma^2) \right\} (1 + \beta^2)^{-3}. \tag{48}$$

The surface is an even function of β and $\Gamma = \cos 3\gamma$. For $r_0, r_2, r_3 \geq 0$, $\hat{H}$ is positive definite and $\tilde{E}(\beta, \gamma)$ has two degenerate global minima, $(\beta = \sqrt{2}, \gamma = 0)$ and $(\beta = \sqrt{2}, \gamma = \pi/3)$ [or equivalently $(\beta = -\sqrt{2}, \gamma = 0)$], at $\tilde{E} = 0$. $\beta = 0$ is always an extremum, which is a local minimum (maximum) for $r_2 > 4r_0$ $(r_2 < 4r_0)$, at $\tilde{E} = 4r_0$. Additional extremal points include saddle points at $[\beta_1 > 0, \gamma = 0, \pi/3]$, $[\beta_2 > 0, \gamma = \pi/6]$ and a local maximum at $[\beta_3 > 0, \gamma = \pi/6]$. The saddle points, when exist, support a barrier separating the various minima, as seen in Figure 6. For large N, the normal modes involve β and γ vibrations about the respective deformed minima, with frequencies

$$\epsilon_{\beta 1} = \epsilon_{\beta 2} = \frac{8}{3}(r_0 + 2r_2)N^2, \tag{49a}$$

$$\epsilon_{\gamma 1} = \epsilon_{\gamma 2} = 4r_3 N^2. \tag{49b}$$
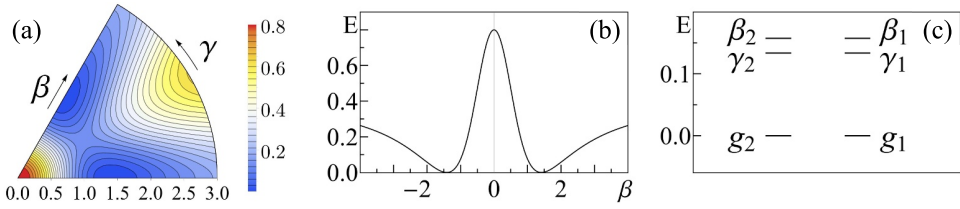
Fig. 6. Prolate-oblate shape coexistence. (a) Contour plots of the energy surface (48), (b) $\gamma = 0$ sections, and (c) bandhead spectrum, for the Hamiltonian $\hat{H}'$ of equation (52), with parameters $r_0 = 0.2$, $r_2 = 0.4$, $r_3 = 0.567$, $C = 0$, $\alpha = 0.018$ and $N = 20$, with multiple SU(3)-PDS and $\overline{\text{SU(3)}}$-PDS. Adapted from [36].

Identifying the collective part with $\hat{C}_2[\text{SO(3)}]$, we arrive at the following complete Hamiltonian,

$$\hat{H}_{\text{PDS}} = r_0\, P_0^\dagger \hat{n}_s P_0 + r_2\, P_0^\dagger \hat{n}_d P_0 + r_3\, W_3^\dagger \cdot \tilde{W}_3 + C\, \hat{C}_2[\text{SO(3)}]. \tag{50}$$

$\hat{H}_{\text{PDS}}$ is not invariant under SU(3) nor $\overline{\text{SU(3)}}$, yet the relations of equations (46) ensure that it has a solvable prolate ground band $g_1(K = 0)$ with good SU(3) symmetry and simultaneously, an oblate ground band, $g_2(\bar{K} = 0)$ with good $\overline{\text{SU(3)}}$ symmetry,

$$g_1 : \ |[N], (\lambda, \mu) = (2N, 0), K = 0, L\rangle \quad L = 0, 2, 4, \ldots, 2N, \tag{51a}$$
$$g_2 : \ |[N], (\bar{\lambda}, \bar{\mu}) = (0, 2N), \bar{K} = 0, L\rangle, \tag{51b}$$
$$E_{g1}(L) = E_{g2}(L) = C\, L(L+1). \tag{51c}$$

$\hat{H}_{\text{PDS}}$ (50) thus exhibits both SU(3)-PDS and $\overline{\text{SU(3)}}$-PDS. It has, however, an undesired feature that it commutes with the $\mathcal{R}_s$ operator of equation (45). Consequently, all non-degenerate eigenstates have well-defined s-parity. This implies vanishing quadrupole moments for an $E2$ operator which is odd under a change of sign of the s-bosons, e.g., for $\chi = 0$ in equation (2). To overcome this difficulty, we add a small s-parity breaking term and consider the Hamiltonian,

$$\hat{H}' = \hat{H}_{\text{PDS}} + \alpha\, \hat{\theta}_2, \tag{52}$$

where $\hat{\theta}_2$ is defined in equation (27a). The added term contributes to $\tilde{E}(\beta, \gamma)$ (48) a small component $\tilde{\alpha}(1+\beta^2)^{-2}[(\beta^2-2)^2+2\beta^2(2-2\sqrt{2}\beta\Gamma+\beta^2)]$, with $\tilde{\alpha} = \alpha/(N-2)$. The linear Γ-dependence distinguishes the two deformed minima and slightly lifts their degeneracy, as well as that of the normal modes (49).

Figure 6 shows $\tilde{E}(\beta, \gamma)$, $\tilde{E}(\beta, \gamma = 0)$ and the bandhead spectrum of $\hat{H}'$ (52). The prolate g_1-band remains solvable with energy $E_{g1}(L)$, equation (51c). The oblate g_2-band experiences a slight shift of order $\frac{32}{9}\alpha N^2$ and displays a rigid-rotor like spectrum. Replacing $\hat{\theta}_2$ by $\bar{\theta}_2 = -\hat{C}_2[\overline{\text{SU(3)}}] + 2\hat{N}(2\hat{N} + 3)$, reverses the sign of the linear Γ term in the energy surface and leads to similar effects, but interchanges the role of prolate and oblate bands in the spectrum of $\hat{H}'$. The SU(3) and $\overline{\text{SU(3)}}$ decompositions in Figure 7 demonstrate that these bands are pure DS basis states, with $(2N, 0)$ and $(0, 2N)$ character, respectively, while excited β and γ bands exhibit considerable mixing. The critical-point Hamiltonian thus has a subset of states with
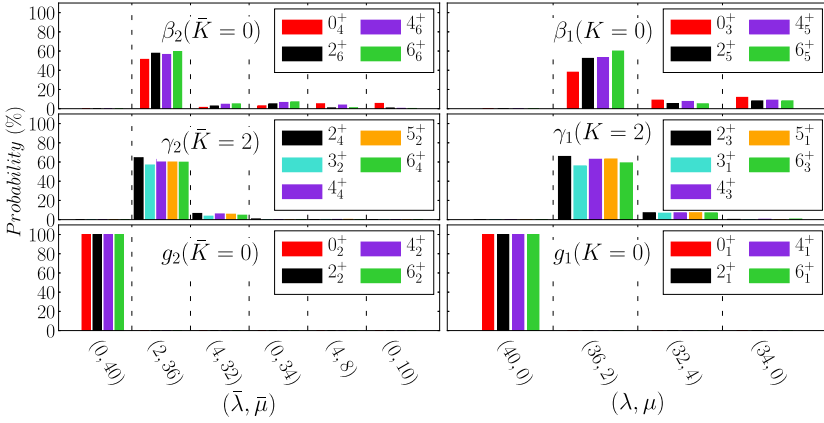
Fig. 7. SU(3) (λ, μ)- and $\overline{\text{SU(3)}}$ $(\bar{\lambda}, \bar{\mu})$-decompositions for members of the prolate (g_1, β_1, γ_1) and oblate (g_2, β_2, γ_2) bands, eigenstates of $\hat{H}'$ (52) with parameters as in Figure 6, resulting in prolate-oblate shape coexistence. Shown are probabilities larger than 5%. Note the purity of the prolate ground band $g_1(K = 0)$ [oblate ground band $g_2(\bar{K} = 0)$] with respect to SU(3) [$\overline{\text{SU(3)}}$]. Adapted from [36].

good SU(3) symmetry, a subset of states with good $\overline{\text{SU(3)}}$ symmetry and all other states are mixed with respect to both SU(3) and $\overline{\text{SU(3)}}$. These are precisely the defining ingredients of SU(3)-PDS coexisting with $\overline{\text{SU(3)}}$-PDS.

Since the wave functions for the members of the g_1 and g_2 bands are known, one can derive closed expressions for their quadrupole moments and $E2$ rates, which are the observables most closely related to the nuclear shapes. For the general $E2$ operator of equation (2), the quadrupole moments are found to be

$$Q_L = \mp e_{\text{b}} \sqrt{\tfrac{16\pi}{40}} \tfrac{L}{2L+3} \left[\tfrac{4(2N-L)(2N+L+1)}{3(2N-1)} (1 \pm \bar{\chi}) \mp \bar{\chi} (4N + 3) \right], \qquad (53)$$

where $\bar{\chi} = \tfrac{2}{\sqrt{7}} \chi$, and the upper (lower) signs correspond to the prolate-g_1 (oblate-g_2) band. The $B(E2)$ values for transitions within these bands, are found to be

$$B(E2; g_i, L + 2 \rightarrow g_i, L)$$
$$= e_{\text{b}}^2 \tfrac{3(L+1)(L+2)}{2(2L+3)(2L+5)} \tfrac{(2N-L)(2N+L+3)}{2} \left[1 - \tfrac{2(N-1)}{3(2N-1)} (1 \pm \bar{\chi}) \right]^2, \qquad (54)$$

where the upper (lower) sign corresponds to $g_1 \rightarrow g_1$ ($g_2 \rightarrow g_2$) transitions. It should be noted that in equations (53)–(54), the factor $(1 \pm \bar{\chi})$ vanishes for $\chi = \mp \tfrac{1}{2} \sqrt{7}$ for which the $E2$ operator coincides with the SU(3) or $\overline{\text{SU(3)}}$ generator $Q^{(2)}$ (13) or $\bar{Q}^{(2)}$ (44).

The purity and good quantum numbers of selected states enable the derivation of symmetry-based selection rules for electromagnetic transitions (notably, for $E2$ and $E0$ decays) and the subsequent identification of isomeric states. The general $E2$ operator $\hat{T}(E2)$ of equation (2), can be written as a sum of $Q^{(2)}$ or $\bar{Q}^{(2)}$ and a $(2, 2)$ tensor under both algebras. The L-states of the g_1 and g_2 bands exhaust, respectively, the $(2N, 0)$ and $(0, 2N)$ irrep of SU(3) and $\overline{\text{SU(3)}}$. Consequently, $Q^{(2)}$ or $\bar{Q}^{(2)}$ cannot contribute to $E2$ transitions between these bands since, as generators, they cannot mix different irreps of the respective algebras. The $(2, 2)$ tensor part of $\hat{T}(E2)$ can
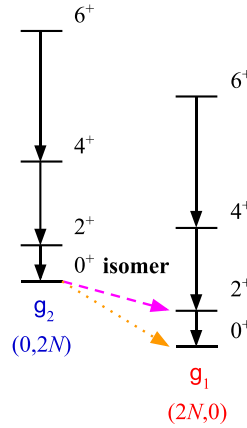
Fig. 8. SU(3) and $\overline{\text{SU(3)}}$ PDSs. Strong intraband $E2$ transitions (solid lines) obey equation (54). Retarded $E2$ and $E0$ decays (dashed and dotted lines) identify isomeric states.

connect the $(2N, 0)$ irrep of g_1 only with the $(2N - 4, 2)$ component in g_2, and the $(0, 2N)$ irrep of g_2 with the $(2, 2N - 4)$ component in g_1, but these components are vanishingly small. These observations imply that interband $(g_2 \leftrightarrow g_1)$ $E2$ transitions, are extremely weak. By similar arguments, $E0$ transitions in-between the g_1 and g_2 bands are extremely weak, since the relevant operator, $\hat{T}(E0) \propto \hat{n}_d$, is a combination of $(0, 0)$ and $(2, 2)$ tensors under both algebras. Accordingly, the $L = 0$ bandhead state of the higher (g_2) band, cannot decay by strong $E2$ or $E0$ transitions to the lower g_1 band, hence, as depicted schematically in Figure 8, displays characteristic features of an isomeric state.

7 Partial dynamical symmetry and proton–neutron shapes

Shell model foundations of the IBM [37] and the observation of mixed-symmetry states in nuclei [38], necessitate the introduction of the interacting proton–neutron boson model [39,40] (named IBM-2 to distinguish it from the original version which retained the name IBM-1). The building blocks of the IBM-2 are monopole and quadrupole bosons, $\{s_\rho^\dagger, d_{\rho,m}^\dagger\}$, of proton type $(\rho = \pi)$ and of neutron type $(\rho = \nu)$, representing pairs of identical valence nucleons. Number conserving bilinear combinations of operators in each set comprise the $U_\rho(6)$ algebra as in Section 1, and bosons of different types commute. Since the separate proton– and neutron–boson numbers, $\hat{N}_\pi$ and $\hat{N}_\nu$, are conserved, the appropriate spectrum generating algebra of the model is $U_\pi(6) \times U_\nu(6)$. Subalgebras can be constructed with the aid of the individual subalgebras, $U_\rho(6)$, $U_\rho(5)$, $SU_\rho(3)$, $\overline{SU(3)_\rho}$, $SO_\rho(6)$, $SO_\rho(5)$, $SO_\rho(3)$. For instance, for a given algebra G_ρ, with generators $\mathcal{G}_\rho$, there is a combined algebra $G_{\pi+\nu}$, with generators $\mathcal{G}_\pi + \mathcal{G}_\nu$.

The dynamical symmetries of the IBM-2 are obtained by identifying the lattices of embedded algebras starting with $U_\pi(6) \times U_\nu(6)$ and ending with the symmetry algebra $SO_{\pi+\nu}(3)$. A typical DS chain in the IBM-2 and related basis have the form [6],

$$U_\pi(6) \times U_\nu(6) \supset G_\pi \times G_\nu \supset G_{\pi+\nu} \supset \ldots \supset SO_{\pi+\nu}(3) \tag{55a}$$

$$|[N_\pi], [N_\nu]; \lambda_\pi, \lambda_\nu, \lambda_{\pi\nu}, \ldots, L\rangle. \tag{55b}$$

The energy surface, $E_{N_\pi, N_\nu}(\beta_\rho, \gamma_\rho, \Omega)$, is now a function of four shape variables $(\beta_\pi, \gamma_\pi, \beta_\nu, \gamma_\nu)$, and three Euler angles (Ω) of the relative orientations between the proton and neutron shapes [41,42]. It is obtained by the expectation value of the Hamiltonian in an intrinsic state,

$$|\beta_\pi, \gamma_\pi, \beta_\nu, \gamma_\nu, \Omega; N_\pi, N_\nu\rangle = |\beta_\pi, \gamma_\pi; N_\pi\rangle|\beta_\nu, \gamma_\nu, \Omega; N_\nu\rangle, \tag{56}$$

which now involves a product of a proton-condensate with deformations (β_π, γ_π), defined as in equation (4), and a rotated (by Ω) neutron condensate with deformations (β_ν, γ_ν). The global minimum of the energy surface, $(\bar{\beta}_\pi, \bar{\beta}_\nu, \bar{\gamma}_\pi, \bar{\gamma}_\nu, \bar{\Omega})$, define the equilibrium proton–neutron shape associated with a given IBM-2 Hamiltonian.

The construction of IBM-2 Hamiltonians with PDS, follows the general algorithm, by considering tensor operators which annihilate subsets of states in particular irreps of the leading subalgebras, $G_\pi \times G_\nu \supset G_{\pi+\nu}$, in the chain (55),

$$\hat{T}_\alpha \, |[N_\pi], [N_\nu]; \lambda_\pi = \Lambda_1, \lambda_\nu = \Lambda_2, \lambda_{\pi\nu} = \Lambda_{12}, \ldots, L\rangle = 0. \tag{57}$$

The PDS Hamiltonian then has the form as in equation (11), $\hat{H}_{PDS} = \hat{H} + \hat{H}_c = \hat{H}_{DS} + \hat{V}_0$, where $\hat{H}$ and $\hat{H}_c$ are the intrinsic and collective parts, equations (7) and (10) respectively, and $\hat{H}_{DS}$ is the DS Hamiltonian for the chain (55).

8 SU(3) PDS and aligned axially-deformed proton–neutron shapes

The DS chain in the IBM-2, related basis and equilibrium deformations, appropriate to the dynamics of aligned prolate-deformed proton–neutron shapes are

$$U_\pi(6) \times U_\nu(6) \supset U_{\pi+\nu}(6) \supset SU_{\pi+\nu}(3) \supset SO_{\pi+\nu}(3), \tag{58a}$$

$$|[N_\pi], [N_\nu]; [N_1, N_2], (\lambda, \mu), K, L\rangle, \quad (\beta_\pi = \beta_\nu = \sqrt{2}, \bar{\gamma}_\pi = \bar{\gamma}_\nu = 0, \bar{\Omega} = 0). \tag{58b}$$

For a given irrep of $U_\pi(6) \times U_\nu(6)$, characterized by N_π and N_ν, the allowed irreps of $U_{\pi+\nu}(6)$ are $[N_1, N_2] = [N_\pi + N_\nu - k, k]$, where $k = 0, 1, \ldots, \min\{N_\pi, N_\nu\}$. Instead of $[N_1, N_2]$ one can use the quantities $N = N_\pi + N_\nu$ and $F = \frac{1}{2}(N_1 - N_2) = \frac{1}{2}N - k$, where (F, N) are the irrep labels of the $SU_F(2) \times U_N(1)$ algebra, which is dual to $U_{\pi+\nu}(6)$. Here $SU_F(2)$ is the F-spin algebra [39], with generators $\hat{F}_+ = s_\pi^\dagger s_\nu + d_\pi^\dagger \cdot \tilde{d}_\nu$, $\hat{F}_- = (\hat{F}_+)^\dagger$, $\hat{F}_0 = (\hat{N}_\pi - \hat{N}_\nu)/2$ and $U_N(1)$ is generated by $\hat{N} = \hat{N}_\pi + \hat{N}_\nu$. The basis states of equation (58) can equivalently be denoted by $|[N_\pi], [N_\nu]; F = \frac{1}{2}N - k, (\lambda, \mu), K, L\rangle$ and have $F_z = \frac{1}{2}(N_\pi - N_\nu)$. States with $N_2 = 0$ ($k = 0$) have maximal F-spin, $F_{max} = \frac{1}{2}N$, and are fully symmetric with respect to the interchange of π and ν bosons, while states with $N_2 \neq 0$ ($k \neq 0$) have $F < F_{max}$, are non-symmetric, and are referred to as mixed-symmetry states.

The linear Casimir of $U_{\pi+\nu}(6)$, $\hat{C}_1[U_{\pi+\nu}(6)] = \hat{N}_\pi + \hat{N}_\nu$, has eigenvalues N, hence no effect on excitation energies. The quadratic Casimir and eigenvalues are given by

$$\hat{C}_2[U_{\pi+\nu}(6)] = \frac{1}{2}\hat{N}(\hat{N} + 8) + 2\hat{F}^2, \tag{59a}$$

$$\langle \hat{C}_2[U_{\pi+\nu}(6)]\rangle = N_1(N_1 + 5) + N_2(N_2 + 3) = \frac{1}{2}N(N + 8) + 2F(F + 1). \tag{59b}$$

The quadratic Casimir of $SU_{\pi+\nu}(3)$, $\hat{C}_2[SU_{\pi+\nu}(3)]$, has the same form as equation (14a), but with $Q^{(2)} = Q_\pi^{(2)} + Q_\nu^{(2)}$ and $L^{(1)} = L_\pi^{(1)} + L_\nu^{(1)}$, and eigenvalues $f_2(\lambda,\mu)$, equation (15a). Similarly, $\hat{C}_2[SO_{\pi+\nu}(3)] = L^{(1)} \cdot L^{(1)}$ with eigenvalues $L(L+1)$.

The DS spectrum associated with the chain (58), consists of rotational bands arranged in $SU(3)$ (λ,μ)-multiplets with prescribed $U_{\pi+\nu}(6)$ (F-spin) symmetry. The lowest states in the symmetric sector $([N,0], F = \frac{1}{2}N)$, involve the ground band with $(\lambda,\mu) = (2N,0)$ and symmetric $\beta_s(K=0)$ and $\gamma_s(K=2)$ bands with $(\lambda,\mu) = (2N-4,2)$. The lowest states in the mixed-symmetry sector $([N-1,1], F = F_{max} - 1 = \frac{1}{2}N - 1)$, involve the scissors $(K=1)$ band with $(\lambda,\mu) = (2N-2,1)$ and the antisymmetric $\beta_a(K=0)$ and $\gamma_a(K=2)$ bands with $(\lambda,\mu) = (2N-4,2)$. The pattern of spectra resembles that expected of rotations and vibrations of a combined axially-deformed shape composed of aligned prolate-deformed π-ν shapes with a common symmetry $\hat{z}$-axis. The energy surface of the Casimir operators of the leading segment in the chain (58), $U_{\pi+\nu}(6) \supset SU_{\pi+\nu}(3)$, has a single minimum at the equilibrium deformations indicated in equation (58b). For these values the condensate $|\bar{\beta}_\rho = \sqrt{2}, \bar{\gamma}_\rho = 0, \bar{\Omega} = 0; N_\pi, N_\nu\rangle$, of equation (56), becomes a lowest weight state in the irrep $(\lambda,\mu) = (2N,0)$ of $SU_{\pi+\nu}(3)$ with maximal F-spin, $F_{max} = N/2$, and serves as the intrinsic state for the ground band $g(K=0)$.

According to the procedure of Section 7, the construction of IBM-2 Hamiltonians with SU(3)-PDS can be accomplished by means of the following operators,

$$P_{\rho,0}^\dagger = d_\rho^\dagger \cdot d_\rho^\dagger - 2(s_\rho^\dagger)^2, \qquad P_{\pi\nu,0}^\dagger = \sqrt{2}\,(d_\pi^\dagger \cdot d_\nu^\dagger - 2s_\pi^\dagger s_\nu^\dagger), \qquad (60a)$$

$$P_{\rho,2m}^\dagger = 2s_\rho^\dagger d_{\rho,m}^\dagger + \sqrt{7}(d_\rho^\dagger d_\rho^\dagger)_m^{(2)}, \; P_{\pi\nu,2m}^\dagger = \sqrt{2}(s_\pi^\dagger d_{\nu,m}^\dagger + s_\nu^\dagger d_{\pi,m}^\dagger) + \sqrt{14}(d_\pi^\dagger d_\nu^\dagger)_m^{(2)}, \; (60b)$$

$$M_{2m}^\dagger = s_\pi^\dagger d_{\nu,m}^\dagger - s_\nu^\dagger d_{\pi,m}^\dagger, \qquad M_{Lm}^\dagger = (d_\pi^\dagger d_\nu^\dagger)_m^{(L)} \; (L=1,3). \qquad (60c)$$

$P_{\rho,Lm}^\dagger$ $(\rho = \pi, \nu)$ are the same $L = 0, 2$ pairs of equation (20). For fixed (Lm), the three operators $P_{i,Lm}^\dagger$ transform as $[2,0]$ under $U_{\pi+\nu}(6)$, i.e., form an F-spin vector with $F = 1$ and $(F_0 = 1, 0, -1) \leftrightarrow (i = \pi, \pi\nu, \nu)$. For fixed i, the set of six operators $(P_{i,0}^\dagger, P_{i,2m}^\dagger)$, span the irrep $(\lambda,\mu) = (0,2)$ of $SU_{\pi+\nu}(3)$. The π-ν boson pairs $M_{Lm}^\dagger$ $(L = 1, 2, 3)$ transform as $[1,1]$ under $U_{\pi+\nu}(6)$, i.e., are F-spin scalars $(F = 0)$, and span the 15-dimensional irrep $(\lambda,\mu) = (2,1)$ of $SU_{\pi+\nu}(3)$. All these operators satisfy

$$P_{i,L'm}|[N_\pi],[N_\nu]; F = \tfrac{1}{2}N, (\lambda,\mu) = (2N,0), K=0, L\rangle = 0, \; N = (N_\pi + N_\nu)/2, \quad (61a)$$

$$M_{i,L'm}|[N_\pi],[N_\nu]; F = \tfrac{1}{2}N, (\lambda,\mu) = (2N,0), K=0, L\rangle = 0, \; L = 0, 2, 4, \ldots, 2N. \; (61b)$$

The indicated L-states span the entire $(2N,0)$ irrep of $SU_{\pi+\nu}(3)$ and form the ground band $g(K=0)$. Equivalently,

$$P_{i,L'm}|g; (2N,0)K=0, F_{max}, F_z; N_\pi, N_\nu\rangle = 0, \; F_{max} = N/2, \qquad (62a)$$

$$M_{L'm}|g; (2N,0)K=0, F_{max}, F_z; N_\pi, N_\nu\rangle = 0, \; F_z = (N_\pi - N_\nu)/2, \qquad (62b)$$

where,

$$|g; (2N,0)K=0, F_{max}, F_z; N_\pi, N_\nu\rangle = |\bar{\beta}_\rho = \sqrt{2}, \bar{\gamma}_\rho = 0, \bar{\Omega} = 0; N_\pi, N_\nu\rangle, \quad (63)$$

is the intrinsic state (56) for the ground band with $SU_{\pi+\nu}(3)$ symmetry $(\lambda,\mu) = (2N,0)$. In general, the conditions $\bar{\beta}_\pi = \bar{\beta}_\nu$, $\bar{\gamma}_\pi = \bar{\gamma}_\nu$ and $\bar{\Omega} = 0$ ensure that the

intrinsic state and the L-states projected from it have good F-spin, $F = F_{max}$ [43]. The relations in equations (61a) and (62a) follow from the fact that the irrep $[N-2]$ of $U_{\pi+\nu}(6)$ does not contain the $SU_{\pi+\nu}(3)$ irreps obtained from the product $(2,0) \times (2N,0)$. Similarly, the relations of equations (61b) and (62b) follow from the fact that $M_{L'm}$ are F-spin scalars, while the $[N-2]$ irrep of $U_{\pi+\nu}(6)$ does not have states with $F = F_{max}$.

An $SU_{\pi+\nu}(3)$ tensor expansion of the Hamiltonian reads,

$$\hat{H} = \sum_{i} \sum_{L=0,2} A_L^{(i)} P_{i,L}^\dagger \cdot \tilde{P}_{i,L} + \sum_{L=1,2,3} B_L M_L^\dagger \cdot \tilde{M}_L + C_2 [P_{\pi\nu,2}^\dagger \cdot \tilde{M}_2 + \text{H.c.}], \quad (64)$$

where H.c. means Hermitian conjugate. Several $SU_{\pi+\nu}(3)$-scalar interactions are contained in the expression (64). Specifically,

$$\hat{\theta}_{\rho,2} = -\hat{C}_2[SU_\rho(3)] + 2\hat{N}_\rho(2\hat{N}_\rho + 3) = P_{\rho,0}^\dagger P_{\rho,0} + P_{\rho,2}^\dagger \cdot \tilde{P}_{\rho,2} \quad (\rho = \pi, \nu), \quad (65a)$$

$$\hat{\theta}_{\pi\nu,2} = -\hat{C}_2[SU_{\pi+\nu}(3)] + 2\hat{N}(2\hat{N} + 3) = \sum_{i} \sum_{L=0,2} P_{i,L}^\dagger \cdot \tilde{P}_{i,L} + 6\mathcal{M}_{\pi\nu}, \quad (65b)$$

$$2\hat{\mathcal{M}}_{\pi\nu} = -\hat{C}_2[U_{\pi+\nu}(6)] + \hat{N}(\hat{N} + 5) = 2[\hat{N}(\hat{N} + 2)/4 - \hat{F}^2]$$
$$= 2(W_2^\dagger \cdot \tilde{W}_2 + 2 \sum_{L=1,3} W_L^\dagger \cdot \tilde{W}_L), \quad (65c)$$

where $\hat{N} = \hat{N}_\pi + \hat{N}_\nu$. The operators $\hat{C}_2[SU_\rho(3)]$ (65a) are defined as in equation (14a), in terms of $Q_\rho^{(2)}$ and $L_\rho^{(1)}$. The operator $\hat{\theta}_{\pi\nu,2}$ (65b), related to $\hat{C}_2[SU_{\pi+\nu}(3)]$, and the Majorana operator (65c), related to $\hat{C}_2[U_{\pi+\nu}(6)]$, are both part of the DS Hamiltonian of the chain (58a). The latter Hamiltonian can be transcribed in the form,

$$\hat{H}_{DS} = A\,\hat{\theta}_{\pi\nu,2} + B\,\hat{\mathcal{M}}_{\pi\nu} + C\,\hat{C}_2[SO_{\pi+\nu}(3)]. \quad (66a)$$
$$E_{DS} = A\,[-f_2(\lambda,\mu) + 2N(2N+3)] + B\,[N(N+2)/4 - F(F+1)]$$
$$+ C\,L(L+1). \quad (66b)$$

In general, the Hamiltonian of equation (64) is not invariant under $SU_{\pi+\nu}(3)$, however, relations (61) ensure that it has a solvable subset of states, with good symmetry, i.e., it has $SU_{\pi+\nu}(3)$ PDS. It has also partial F-spin symmetry [44], since the solvable states have good F-spin quantum number, $F = F_{max}$, but the Hamiltonian is an F-spin scalar only when $A_L^{(\pi)} = A_L^{(\nu)} = A_L^{(\pi\nu)}$ ($L = 0,2$) and $C_2 = 0$.

The operators $P_{i,L'm}^\dagger$, equations (60a)–(60b), satisfy also

$$P_{i,L'm}|[N_\pi], [N_\nu]; F = \tfrac{1}{2}N - 1, (\lambda,\mu) = (2N-2,1), K = 1, L\rangle = 0, \quad (67a)$$
$$P_{i,L'm}|sc; (2N-2,1)K = 1, F_{max}-1, F_z; N_\pi, N_\nu\rangle = 0. \quad (67b)$$

In equation (67a), the states $L = 1, 2, \ldots, 2N-1$ span the entire $(2N-2,1)$ irrep of $SU_{\pi+\nu}(3)$ and form the scissors band $sc(K=1)$, whose intrinsic state,

$$|sc; (2N-2,1)K = 1, F_{max}-1, F_z; N_\pi, N_\nu\rangle$$
$$= \tfrac{1}{\sqrt{N}} \Gamma_{sc}^\dagger |g; F_{max}-1, F_z; N_\pi - 1, N_\nu - 1\rangle, \quad (68a)$$
$$\Gamma_{sc}^\dagger = b_{c,\pi}^\dagger d_{\nu,1}^\dagger - d_{\pi,1}^\dagger b_{c,\nu}^\dagger, \quad b_{c,\rho}^\dagger = \tfrac{1}{\sqrt{3}}(\sqrt{2}\,d_{\rho,0}^\dagger + s_\rho^\dagger), \quad (68b)$$

is a lowest weight state of this irrep with $F = F_{max} - 1$. Here $|g; F_{max}-1, F_z; N_\pi -1, N_\nu -1\rangle$ is obtained from equation (63) and $\Gamma_{sc}^\dagger$ is an F-spin scalar. Equations (67) follow from the fact that the $[N-2,0]$ and $[N-3,1]$ irreps of $U_{\pi+\nu}(6)$ do not contain the $SU_{\pi+\nu}(3)$ irreps obtained from the product $(2,0) \times (2N-2,1)$. These relations ensure that the states of the ground and scissors bands are solvable eigenstates of the Hamiltonian,

$$\hat{H}_{PDS} = \hat{H}_{DS} + \sum_i \sum_{L=0,2} A_L^{(i)} P_{i,L}^\dagger \cdot \tilde{P}_{i,L} + B\hat{\mathcal{M}}_{\pi\nu} + C\,\hat{C}_2[SO_{\pi+\nu}(3)], \qquad (69)$$

where $\hat{H}_{DS}$ is given in equation (66). Furthermore, the operators $P_{i,L=0}^\dagger$ satisfy

$$P_{i,0}|[N_\pi],[N_\nu]; F = \tfrac{1}{2}N, (\lambda,\mu) = (2N-4,2), K = 2, L\rangle = 0, \qquad (70a)$$

$$P_{i,0}|[N_\pi],[N_\nu]; F = \tfrac{1}{2}N-1, (\lambda,\mu) = (2N-4,2), K = 2, L\rangle = 0. \qquad (70b)$$

In equations (70a) and (70b), the states have $L = 2, 3, \ldots, 2N-2$ and span part of the $[(2N-4,2), F = F_{max}]$ and $[(2N-4,2), F = F_{max}-1]$ irreps of $SU_{\pi+\nu}(3)$. They form the symmetric $\gamma_s(K=2)$ and antisymmetric $\gamma_a(K=2)$ gamma bands, respectively. These properties follow from the fact that $P_{i,0}$ annihilate the intrinsic states of these bands [45],

$$|\gamma_s; (2N-4,2)K = 2, F_{max} = \tfrac{1}{2}N, F_z; N_\pi, N_\nu\rangle$$
$$= \sum_q C_{1,q;F_{max},F_z-q}^{F_{max},F_z} B_{\gamma2;1q}^\dagger |g; F_{max}-1, F_z-q; N_\pi-1-q, N_\nu-1+q\rangle, \quad (71a)$$

$$|\gamma_a; (2N-4,2)K = 2, F_{max} - 1 = \tfrac{1}{2}N - 1, F_z; N_\pi, N_\nu\rangle$$
$$= \sum_q C_{1,q;F_{max},F_z-q}^{F_{max}-1,F_z} B_{\gamma2;1q}^\dagger |g; F_{max}-1, F_z-q; N_\pi-1-q, N_\nu-1+q\rangle, \quad (71b)$$

where $B_{\gamma2;F=1,q}^\dagger = \frac{1}{3\sqrt{2}} P_{i;2,2}^\dagger$ $(q = 1,0,-1 \leftrightarrow i = \pi, \pi\nu, \nu)$. Here $C_{1,q;F,F_z-q}^{F,F_z}$ is a Clebsch Gordan coefficient and $F_z = \tfrac{1}{2}(N_\pi - N_\nu)$. Altogether, equations (61), (67) and (70) ensure that the following Hamiltonian,

$$\hat{H}_{PDS} = \hat{H}_{DS} + \sum_i A_0^{(i)} P_{i,0}^\dagger P_{i,0} + B\hat{\mathcal{M}}_{\pi\nu} + C\,\hat{C}_2[SO_{\pi+\nu}(3)], \qquad (72)$$

has a number of solvable bands including the ground (g), scissors (sc), symmetric-gamma (γ_s) and anti-symmetric gamma (γ_a) bands,

$$g: \ |[N_\pi],[N_\nu]; F = \tfrac{1}{2}N, (\lambda,\mu) = (2N,0), K = 0, L\rangle$$
$$E_g(L) = CL(L+1), \qquad (73a)$$

$$sc: \ |[N_\pi],[N_\nu]; F = \tfrac{1}{2}N-1, (\lambda,\mu) = (2N-2,1), K = 1, L\rangle$$
$$E_{sc}(L) = (6A+B)N + CL(L+1), \qquad (73b)$$

$$\gamma_s: \ |[N_\pi],[N_\nu]; F = \tfrac{1}{2}N, (\lambda,\mu) = (2N-4,2), K = 2, L\rangle$$
$$E_{\gamma_s}(L) = 6A(2N-1) + CL(L+1), \qquad (73c)$$

$$\gamma_a: \ |[N_\pi],[N_\nu]; F = \tfrac{1}{2}N-1, (\lambda,\mu) = (2N-4,2), K = 2, L\rangle$$
$$E_{\gamma_a}(L) = 6A(2N-1) + BN + CL(L+1). \qquad (73d)$$

Table 2. Number of interactions in the IBM-1 and IBM-2 for the general Hamiltonians, the SU(3) DS (12) and $SU_{\pi+\nu}(3)$ DS (58) limits, and their PDS extensions. On the left of $\mapsto$ is the number of interactions of a given order; this reduces to the number on the right of $\mapsto$ if one is only interested in excitation energies in a single nucleus.

Order	Number of interactions IBM-1		
	General	SU(3) DS	SU(3) PDS
1	$2 \mapsto 1$	$1 \mapsto 0$	$1 \mapsto 0$
2	$7 \mapsto 5$	$3 \mapsto 2$	$4 \mapsto 3$
3	$17 \mapsto 10$	$4 \mapsto 1$	$10 \mapsto 6$
$1+2+3$	$26 \mapsto 16$	$8 \mapsto 3$	$15 \mapsto 9$
Order	Number of interactions IBM-2		
	General	$SU_{\pi+\nu}(3)$ DS	$SU_{\pi+\nu}(3)$ PDS
1	$4 \mapsto 2$	$2 \mapsto 0$	$2 \mapsto 0$
2	$25 \mapsto 18$	$6 \mapsto 3$	$14 \mapsto 11$
$1+2$	$29 \mapsto 20$	$8 \mapsto 3$	$16 \mapsto 11$

Explicit expressions for electromagnetic rates of $E2$ and $M1$ transitions among these solvable states, are available [46].

9 Concluding remarks

We have considered several classes of two- and three-body IBM Hamiltonians with SU(3) PDS, for single-, multiple-, and proton–neutron shapes. In each class, subsets of solvable states with good symmetry allow for analytic expressions of energies and E2 rates, as well as, symmetry-based selection rules. The cases considered include: (a) IBM-1 Hamiltonians with SU(3)-PDS, describing the dynamics of a single prolate-deformed shape, with solvable ground $g(K = 0)$ and $\gamma^k(K = 2k)$ bands. (b) IBM-1 Hamiltonians with SU(3)-PDS, describing the dynamics of a single prolate-deformed shape, with solvable ground $g(K = 0)$ and $\beta(K = 0)$ bands. (c) IBM-1 Hamiltonians with simultaneous SU(3) and $\overline{SU(3)}$ PDSs, describing the dynamics of coexisting prolate and oblate shapes, with solvable ground bands. (d) IBM-2 Hamiltonians with $SU_{\pi+\nu}(3)$ PDS, describing the dynamics of aligned prolate-deformed π-ν shapes, with solvable symmetric states (maximal F-spin) comprising the ground and symmetric-gamma bands, and solvable mixed-symmetry states (non-maximal F-spin), comprising the scissors and anti-symmetric gamma bands.

The analysis serves to highlight the merits gained by using the notion of PDS. (i) PDS allows more flexibility by relaxing the constraints of an exact DS and, therefore, can be applied to more realistic situations. (ii) PDS supports a subset of solvable states with good symmetry, for which related observables can be calculated analytically and often lead to parameter-free predictions that can be tested by comparing with the experimental data. This was demonstrated in Figure 2, for $\gamma \to g$ B(E2) ratios in ^{168}Er. (iii) PDS picks particular symmetry-breaking terms which on one hand, do not destroy results previously obtained with a DS for a segment of the spectrum, but on the other hand, allow deviations from undesired DS predictions for the non-solvable mixed states. This was exemplified by the improved PDS description of the staggering in the γ-band of ^{156}Gd, shown in Figure 4, without harming the good DS description of $E2$ transitions involving states of the ground and beta bands, shown in Table 1. (iv) PDS can be generalized to encompass several incompatible PDSs, hence provide symmetry-based benchmarks for the study of shape-coexistence

in nuclei. This was illustrated for multiple prolate-oblate shapes in Section 6. (v) PDS provides a selection criterion for many-body interactions. This is particularly important in the presence of higher-order terms (e.g., three-body) or more degrees of freedom (e.g., protons and neutrons), for which the number of possible interactions in the effective Hamiltonian grows rapidly and a selection criterion is called for.

To illustrate the increase in flexibility of a Hamiltonian with SU(3) PDS, we list in Table 2 the number of interactions under the different scenarios. Up to third order, a general rotationally invariant IBM-1 Hamiltonian has 26 independent interactions, decreasing to 16 if one is only interested in excitation energies in a single nucleus. (This excludes terms involving $\hat{N}$). An IBM-1 Hamiltonian with SU(3) DS has, up to third order, 8 independent terms but 5 of them ($\hat{N}$, $\hat{N}^2$, $\hat{N}^3$, $\hat{N}\hat{L}^2$, and $\hat{N}\hat{C}_2[\mathrm{SU}(3)]$) are constant in a single nucleus or can be absorbed in an interaction of lower order, leaving only the 3 genuinely independent terms shown in equation (28a). The corresponding numbers for an IBM-1 Hamiltonian with SU(3) PDS are 15 and 9. The latter number agrees with the 9 terms in the Hamiltonian (29). In the IBM-2 with one- and two-body terms, the general Hamiltonian has 29 independent interactions, decreasing to 20 relevant for excitation energies in a single nucleus. (This excludes terms involving $\hat{N}_\pi$ and $\hat{N}_\nu$). An IBM-2 Hamiltonian with the $\mathrm{SU}_{\pi+\nu}(3)$ DS of equation (58), has 8 independent terms decreasing to 3 for excitation energies, upon excluding $\hat{N}_\rho$, $\hat{N}_\rho^2$ and $\hat{N}_\pi N_\nu$. The corresponding numbers for an IBM-2 Hamiltonian with $\mathrm{SU}_{\pi+\nu}(3)$ PDS are 16 and 11. The latter number agrees with the 10 parameters in equation (64), plus the $\mathrm{C}_2[\mathrm{SO}_{\pi+\nu}(3)]$ term of $\hat{H}_{\mathrm{DS}}$ (66). We conclude from Table 2 that more than half of all possible interactions in the IBM-1 and in the IBM-2, have in fact an SU(3) PDS.

In the present contribution, we have focused on bosonic SU(3)-type of PDSs. However, the PDS notion is not restricted to a specific algebra nor type of constituent particles (bosons and fermions). Examples of PDSs associated with other DS chains of the IBM are known [16,34,36,44,47–50], as well as in the interacting boson-fermion model of odd-mass nuclei [51] and in purely fermionic models of nuclei [52–58]. In addition to applications to nuclear spectroscopy, Hamiltonians with PDS are also relevant to studies of mixed systems with coexisting regularity and chaos [59–61].

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References

1. J.P. Elliott, Proc. Roy. Soc. A **245**, 128 (1958)
2. J.P. Elliott, Proc. Roy. Soc. A **245**, 562 (1958)
3. D.J. Rowe, Rep. Prog. Phys. **48**, 1419 (1985)
4. R.D. Ratna-Raju, K.T. Hecht, B.D. Chang, J.P. Draayer, Nucl. Phys. A **202**, 433 (1973)
5. J.N. Ginocchio, Ann. Phys. (NY) **126**, 234 (1980)
6. F. Iachello, A. Arima, *The interacting boson model* (Cambridge University Press, Cambridge, 1987)
7. F. Iachello, P. Van Isacker, *The interacting boson-fermion model* (Cambridge University Press, Cambridge, 1991)
8. F. Iachello, *Lie algebras and applications* (Springer-Verlag, Berlin, 2015)
9. R. Gilmore, J. Math. Phys. **20**, 891 (1979)

10. For a review see, W.M. Zhang, D.H. Feng, R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990)
11. J.N. Ginocchio, M.W. Kirson, Phys. Rev. Lett. **44**, 1744 (1980)
12. A.E.L. Dieperink, O. Scholten, F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980)
13. For a review see, A. Leviatan, Prog. Part. Nucl. Phys. **66**, 93 (2011)
14. V.K.B. Kota, *SU(3) symmetry in atomic nuclei* (Springer Nature, Singapore, 2020)
15. Y. Alhassid, A. Leviatan, J. Phys. A **25**, L1265 (1992)
16. J.E. García-Ramos, A. Leviatan, P. Van Isacker, Phys. Rev. Lett. **102**, 112502 (2009)
17. M.W. Kirson, A. Leviatan, Phys. Rev. Lett. **55**, 2846 (1985)
18. A. Leviatan, Ann. Phys. (NY) **179**, 201 (1987)
19. H. De Meyer, G. Vanden Berghe, J. Van der Jeugt, J. Math. Phys. **26**, 3109 (1985)
20. G. Vanden Berghe, H.E. De Meyer, P. Van Isacker, Phys. Rev. C **32**, 1049 (1985)
21. D. Bonatsos, Phys. Lett. B **200**, 1 (1985)
22. J. Vanthournout, Phys. Rev. C **41**, 2380 (1990)
23. G. Rosensteel, J.P. Draayer, K.J. Weeks, Nucl. Phys. A **419**, 1 (1984)
24. J.P. Draayer, G. Rosensteel, Nucl. Phys. A **439**, 61 (1985)
25. A. Leviatan, Phys. Rev. Lett. **77**, 818 (1996)
26. A. Leviatan, I. Sinai, Phys. Rev. C **60**, 061301 (1999)
27. P. Van Isacker, Phys. Rev. C **27**, 2447 (1983)
28. R.F. Casten, R.B. Cakirli, K. Blaum, A. Couture, Phys. Rev. Lett. **113**, 112501 (2014)
29. A. Couture, R.F. Casten, R.B. Cakirli, Phys. Rev. C **91**, 014312 (2015)
30. R.F. Casten, J. Jolie, R.B. Cakirli, A. Couture, Phys. Rev. C **94**, 061303(R) (2016)
31. A. Leviatan, J.E. Garca-Ramos, P. Van Isacker, Phys. Rev. C **87**, 021302(R) (2013)
32. A. Aprahamian, R.C. de Haan, S.R. Lesher, C. Casarella, A. Stratman, H.G. Börner, H. Lehmann, M. Jentschel, A.M. Bruce, Phys. Rev. C **98**, 034303 (2018)
33. For a review see, K. Heyde, J.L. Wood, Rev. Mod. Phys. **83**, 1467 (2011)
34. A. Leviatan, Phys. Rev. Lett. **98**, 242502 (2007)
35. A. Leviatan, D. Shapira, Phys. Rev. C **93**, 051302(R) (2016)
36. A. Leviatan, N. Gavrielov, Phys. Scr. **92**, 114005 (2017)
37. For a review see, F. Iachello, I. Talmi, Rev. Mod. Phys. **59**, 339 (1987)
38. For a review see, K. Heyde, P. von Neumann Cosel, A. Richter, Rev. Mod. Phys. **82**, 2365 (2010)
39. A. Arima, T. Otsuka, F. Iachello, I. Talmi, Phys. Lett. B **66**, 205 (1977)
40. T. Otsuka, A. Arima, F. Iachello, I. Talmi, Phys. Lett. B **76**, 139 (1978)
41. A. Leviatan, M.W. Kirson, Ann. Phys. (NY) **201**, 13 (1990)
42. J.N. Ginocchio, A. Leviatan, Ann. Phys. (NY) **216**, 152 (1992)
43. A. Leviatan, J.N. Ginocchio, M.W. Kirson, Phys. Rev. Lett. **65**, 2853 (1990)
44. A. Leviatan, J.N. Ginocchio, Phys. Rev. C **61**, 024305 (2000)
45. H.C. Wu, A.E. Dieperink, S. Pittel, Phys. Rev. C **34**, 703 (1986)
46. P. Van Isacker, K. Heyde. J. Jolie, A. Sevrin, Ann. Phys. (NY) **171**, 253 (1986)
47. A. Leviatan, P. Van Isacker, Phys. Rev. Lett. **89**, 222501 (2002)
48. C. Kremer, J. Beller, A. Leviatan, N. Pietralla, G. Rainovski, R. Trippel, P. Van Isacker, Phys. Rev. C **89**, 041302(R) (2014)
49. A. Leviatan, N. Gavrielov, J.E. Garca-Ramos, P. Van Isacker, Phys. Rev. C **98**, 031302(R) (2018)
50. I. Talmi, Phys. Lett. B **405**, 1 (1997)
51. P. Van Isacker, J. Jolie, T. Thomas, A. Leviatan, Phys. Rev. C **92**, 011301(R) (2015)
52. J. Escher, A. Leviatan, Phys. Rev. Lett. **84**, 1866 (2000)
53. J. Escher, A. Leviatan, Phys. Rev. C **65**, 054309 (2002)
54. D.J. Rowe, G. Rosensteel, Phys. Rev. Lett. **87**, 172501 (2001)
55. G. Rosensteel, D.J. Rowe, Phys. Rev. C **67**, 014303 (2003)
56. P. Van Isacker, S. Heinze, Phys. Rev. Lett. **100**, 052501 (2008)
57. I. Talmi, Nucl. Phys. A **846**, 31 (2010)
58. P. Van Isacker, S. Heinze, Ann. Phys. **349**, 73 (2014)
59. N. Whelan, Y. Alhassid, A. Leviatan, Phys. Rev. Lett. **71**, 2208 (1993)
60. A. Leviatan, N.D. Whelan, Phys. Rev. Lett. **77**, 5202 (1996)
61. M. Macek, A. Leviatan, Ann. Phys. (NY) **351**, 302 (2014)