האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

Advanced cosmology course

GRAVITATIONAL INSTABILITIES

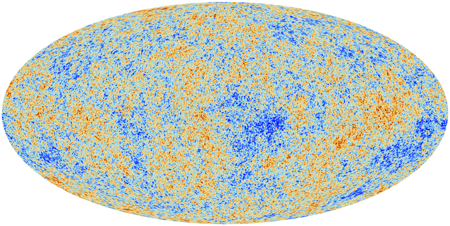
Jonathan Freundlich

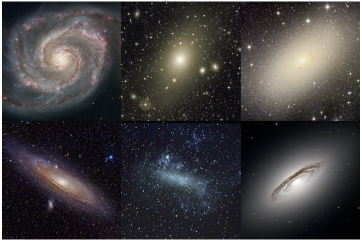
Outline

Outline

Galaxy formation in the ΛCDM model

From a very homogeneous early Universe to the current distribution of galaxies, clusters and voids...





- The standard ΛCDM cosmological model:
  - 26% cold dark matter (CDM)
  - 5% baryons (ordinary matter)
  - 69% dark energy (accelerated expansion, Λ)
- The Universe is initially very homogeneous (cf. cosmic microwave background, 380 000 years after the Big Bang).
- Gravitational attraction vs. the expansion of the Universe.
- Hierarchical dark matter dynamics, baryons cool and contract within dark matter haloes.

Images: ESA/Planck collaboration/C. Mihos/ESO/A. Block/NOAO/AURA/NSF/ A. Evans/NASA/S. Beckwith/Hubble Heritage Team/STScI/AURA/Skatebiker

Volker Springel et al. (2008)

Star formation

Stars form from cold giant molecular gas clouds in the interstellar medium:

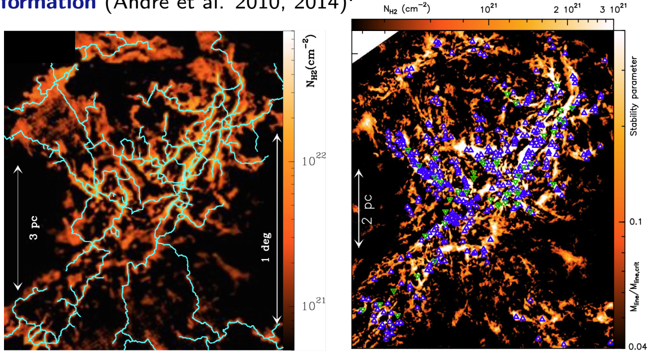
- Mostly composed of hydrogen, masses of $10^5 - 10^7\ \mathrm{M}_\odot$, sizes over a few tens of parsec
- Gravitational collapse, fragmentation into high-density cores
- Inside the pre-stellar cores, temperature and pressure rise
- Nuclear fusion reactions and stellar nucleosynthesis

$(1\ \mathrm{parsec} = 3.09\ 10^{16}\mathrm{m})$

The Orion nebula, a stellar nursery (NASA)

Substructures of the interstellar medium

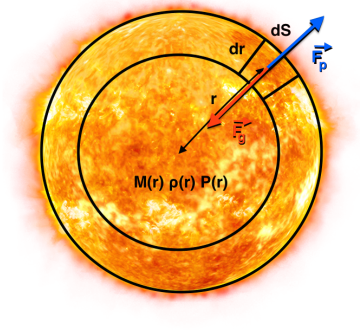
- **Giant molecular clouds host complex networks of filamentary structures.**
  - lengths of a few parsecs, ~ 0.1 parsec wide (Arzoumanian et al. 2011)
  - large-scale interstellar turbulence, magnetic fields & gravitational instabilities
  - they seem to precede star formation
- **A majority of pre-stellar cores lie within supercritical filaments:** 75% in the Aquila complex (Konyves et al. 2015)
- **The formation of turbulence-driven filaments may represent the first step towards core and star formation** (André et al. 2010, 2014):



P. André & D. Arzoumanian, Konyves et al. 2015

Stellar equilibrium

- **Inside a star, the inward gravitational force is balanced by pressure:**
  - thermal pressure
  - radiation pressure from the inner core
  - electron degeneracy pressure (at high density)
- **Self regulation:** When the fusion rate increases, temperature and pressure increase, the core expands, hence temperature and pressure decrease, and the fusion rate decreases
- **Stellar evolution:** When hydrogen fusion reactions stop at the center, the pressure support decrease and the core starts contracting. The energy dissipated through radiation expels the outer layers and helium fusion reactions ignite: the star becomes a red giant.



Hydrostatic equilibrium:

Gravitation $F_g = -\frac{GM(r)\Delta M}{r^2}$ with $\Delta M = \rho(r) dr dS$

Pressure $F_p = -\left.\frac{dP}{dr}\right|_r drdS$

$$\vec{F_p} + \vec{F_g} = \vec{0} \Rightarrow \left.\frac{dP}{dr}\right|_r = -\frac{GM(r)}{r^2}\rho(r)$$

For a uniform density ρ:

$M(r) = \frac{4\pi G\rho}{3} r^3$

$P(r) = P_0 - \frac{2\pi G \rho^2}{3} r^2$

$P = 0$ at the star radius $R \Rightarrow M = \left(\frac{6P_0^3}{\pi G^3 \rho^4}\right)^{1/2}$

High density stars usually have smaller masses

Outline

Gravitational collapse without pressure support

(This calculation could apply to a cloud of dust without pressure, for example)

We consider a cloud initially at rest with **uniform density** ρ_0, **total mass** M_0 and **radius** R_0.

The cloud collapses following

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

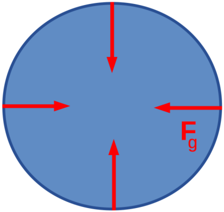
where M is the mass within radius r, which is a constant through the collapse.

$$v^2 = \frac{2GM}{r} - \frac{2GM}{R_0}$$

$$dt = \frac{dr}{\sqrt{2GM\left(\frac{1}{r} - \frac{1}{R_0}\right)}}$$

So the **free-fall time** (or dynamical time) is:

$$t_{ff} = \int_0^{R_0} \frac{dr}{\sqrt{2GM\left(\frac{1}{r} - \frac{1}{R_0}\right)}} = \sqrt{\frac{3\pi}{32G\rho_0}}$$



The free-fall time is independent of R_0.

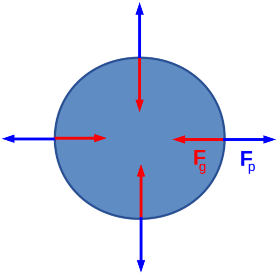
The Jeans instability: pressure vs. gravity

- **Orders of magnitude:**

$$F_g \propto \frac{GM_0^2}{R_0^2} \propto G\rho_0^2 R_0^4$$

$$F_p \propto pR_0^2 \propto c_0^2 \rho_0 R_0^2$$

where c_0 is the sound speed ($p = c_0^2 \rho$ for an ideal gas of adiabatic index $\gamma = 1$).



Gravitational collapse when $F_g > F_p \Leftrightarrow \frac{R_0}{c_0} > \frac{1}{\sqrt{G\rho_0}}$:
big, dense, cold gas clouds are unstable.

- **Interpretation in terms of timescales:**

$1/\sqrt{G\rho_0}$ is the free-fall time

R_0/c_0 is the sound crossing time

The sound crossing time is the time needed for sound waves to cross the region and attempt to push back and re-establish the pressure equilibrium.

The Jeans instability: local perturbation

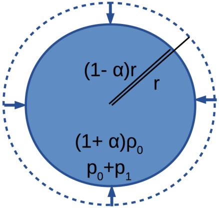
(Binney & Tremaine 2008, section 5.2, p. 401)

▶ Local compression

We consider a fluid of density ρ_0 and pressure p_0, initially at rest. If a sphere of radius r is compressed to a radius $(1-\alpha)r$ where $\alpha \ll 1$, the resulting sphere has increased density and pressure

$$\rho_0 + \rho_1 \approx (1+\alpha)\rho_0$$

$$p_0 + p_1 \approx p_0 + \frac{\partial p}{\partial \rho}\rho_1 \approx p_0 + c_0^2 \alpha \rho_0$$



Additional pressure force per unit mass: $F_{p1} \approx \frac{p_1}{r\rho_0} \approx \alpha c_0^2/r$, as $\vec{F_p} = \frac{1}{\rho}\vec{\nabla} p$

Additional gravitational force per unit mass: $F_{g1} \approx \alpha \frac{GM}{r^2} \approx \alpha G \rho_0 r$, with $M = \frac{4\pi}{3}\rho_0 r^3$.

▶ Fate of the perturbation:

If $F_{g1} < F_{p1}$, the net additional force is outward, the fluid re-expands and the perturbation is stable.

If $F_{g1} > F_{p1}$, the net additional force is inward, the fluid continues to contract and the perturbation is unstable. This corresponds to $\boxed{G\rho_0 r > c_0^2/r \Leftrightarrow \frac{r}{c_0} > \frac{1}{\sqrt{G\rho_0}}}$.

▶ **Jeans length:** perturbations with scale larger than $\lambda_{\rm Jeans} \approx c/\sqrt{G\rho_0}$ are unstable.

The Jeans instability: linear perturbations

- **Hydrodynamic equations:**

For a barotropic fluid (i.e., in which pressure is only a function of density, h being the specific enthalpy):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi \text{ (Euler's equation)}$$
$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0 \text{ (mass conservation)}$$
$$\nabla^2\Phi = 4\pi G\rho \text{ (Poisson equation)}$$
$$\frac{1}{\rho}\vec{\nabla}p = \vec{\nabla}h \text{ (equation of state)}$$

- **Small perturbations:**

First order perturbations to the an unperturbed barotropic fluid:

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \rho_1(\vec{r},t),\ \rho_1 \ll \rho_0$$
$$h(\vec{r},t) = h_0(\vec{r}) + h_1(\vec{r},t),\ h_1 \ll h_0$$
$$\Phi(\vec{r},t) = \Phi_0(\vec{r}) + \Phi_1(\vec{r},t),\ \Phi_1 \ll \Phi_0$$
$$\vec{v}(\vec{r},t) = \vec{v_0}(\vec{r}) + \vec{v_1}(\vec{r},t)$$

- **Linear equations** governing the response to the perturbations:

$$\begin{cases} \dfrac{\partial \vec{v_1}}{\partial t} + (\vec{v_0}.\vec{\nabla})\vec{v_1} + (\vec{v_1}.\vec{\nabla})\vec{v_0} = -\vec{\nabla}(h_1 + \Phi_1) \\ \dfrac{\partial \rho_1}{\partial t} + \vec{\nabla}.(\rho_0\vec{v_1}) + \vec{\nabla}.(\rho_1\vec{v_0}) = 0 \\ \nabla^2\Phi_1 = 4\pi G\rho_1 \\ h_1 = c_0^2\dfrac{\rho_1}{\rho_0}, \end{cases}$$

where the speed of sound in the uniform medium is defined as $c_0^2 = \left(\dfrac{dp}{d\rho}\right)_{\rho_0}$.

The Jeans instability: linear perturbations

- **Jean's swindle:**

For an infinite, homogeneous, unperturbed system, the hydrodynamic equations can not be verified simultaneously: Poisson's equation requires $\nabla^2\Phi_0 = 4\pi\rho_0$ but $\vec{\nabla}\Phi_0 = 0$ for a uniform gravitational potential and $\rho_0 \neq 0$. We assume that Poisson's equation properly describes the relation between the **perturbed** density and potential, while some unspecified source cancels the unperturbed term in $\nabla^2\Phi_0$.

- **Linear equations** in the case of an infinite, immobile, unperturbed medium:

$$\begin{cases} \dfrac{\partial \vec{v_1}}{\partial t} = -\vec{\nabla}(h_1 + \Phi_1) \\ \dfrac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla}.\vec{v_1} = 0 \\ \nabla^2\Phi_1 = 4\pi G\rho_1 \\ h_1 = c_0^2 \dfrac{\rho_1}{\rho_0}. \end{cases}$$

Considering normal modes $\propto e^{-i\omega t + \vec{k}.\vec{r}}$:

$$\begin{cases} -i\omega\vec{v_1} = -i\vec{k}(c_0^2 \dfrac{\rho_1}{\rho_0} + \Phi_1) \\ -i\omega\rho_1 + i\rho_0\vec{k}.\vec{v_1} = 0 \\ -k^2\Phi_1 = 4\pi G\rho_1. \end{cases}$$

- **Dispersion relation:** $\boxed{\omega^2 = c_0^2 k^2 - 4\pi G\rho_0.}$

• For wavenumbers k above $k_{\rm Jeans} = \sqrt{4\pi G\rho_0/c_0^2}$, $\omega^2 > 0$ and the solutions are oscillatory

• For wavenumbers below $k_{\rm Jeans}$, the solutions can be exponentially growing: the system is gravitationally unstable.

Perturbations of size larger than the Jeans length $\lambda_{\rm Jeans} = \sqrt{\pi c_0^2/G\rho_0}$ are amplified while smaller perturbations fade away.

The Jeans instability: an idealized situation

- **Limits of the Jeans instability:**
  - spherical
  - no rotation
  - no magnetic fields
  - no turbulence

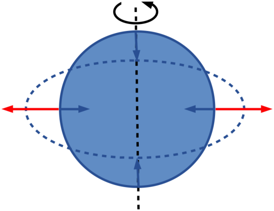
Also, in the previous calculations, we did not take stars into account: this would require distribution functions and the Boltzmann statistical equation.

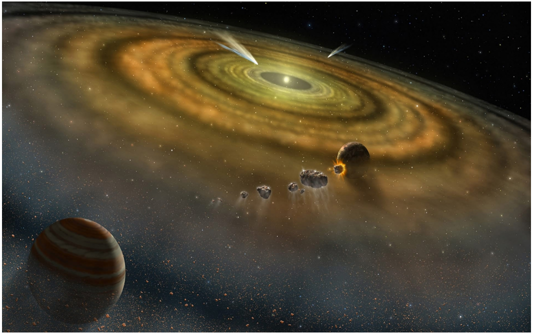
- **Simulation example: formation of stars in a turbulent gas core (Banerjee et al.):**

Outline

Can galactic or proto-planetary disks be gravitationally unstable?

- **Disks form naturally when there is rotation**
  - Centrifugal force
  - Coriolis force
  - Flat geometry





Credits: NASA, ESA, S. Beckwith (STScI), and the Hubble Heritage Team STScI/AURA; NASA/FUSE/Lynette Cook

The Toomre instability: linear perturbations

(Binney & Tremaine 2008, section 6.2.2.c, p. 488)

- **Assumptions:**
  - Cylindrical coordinates (R, ϕ, z)
  - Flat disk: $\rho \to \Sigma$, $z = 0$
  - Uniform unperturbed surface density Σ_0
  - $\vec{v_0} = R\Omega_0 \vec{u_\phi}$ (solid rotation)

- **Hydrodynamic equations:**

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\vec{\nabla}(\Phi + h) \text{ (Euler's equation)}$$
$$\frac{\partial \Sigma}{\partial t} + \vec{\nabla}.(\Sigma\vec{v}) = 0 \text{ (mass conservation)}$$
$$\nabla^2\Phi = 4\pi G\Sigma\delta(z) \text{ (Poisson equation)}$$

- **Small axisymmetric perturbations:**

$$\Sigma(\vec{r}, t) = \Sigma_0 + \Sigma_1(R, t),\ \rho_1 \ll \rho_0$$
$$h(\vec{r}, t) = h_0(\vec{r}) + h_1(R, t),\ h_1 \ll h_0$$
$$\Phi(\vec{r}, t) = \Phi_0(\vec{r}) + \Phi_1(R, z, t),\ \Phi_1 \ll \Phi_0$$
$$v_R(\vec{r}, t) = v_{R1}(R, t)$$
$$v_\phi(\vec{r}, t) = R\Omega_0 + v_{\phi_1}(R, t)$$

- **Linear equations** governing the response to the perturbations:

$$\begin{cases} \dfrac{\partial \vec{v_1}}{\partial t} + (\vec{v_0}.\vec{\nabla})\vec{v_1} + (\vec{v_1}.\vec{\nabla})\vec{v_0} = -\vec{\nabla}(h_1 + \Phi_1) \\ \dfrac{\partial \Sigma_1}{\partial t} + \vec{\nabla}.(\Sigma_0\vec{v_1}) + \vec{\nabla}.(\Sigma_1\vec{v_0}) = 0 \\ \nabla^2\Phi_1 = 4\pi G\Sigma_1\delta(z) \\ h_1 = c_0^2 \dfrac{\Sigma_1}{\Sigma_0} \end{cases}$$

Linearized equation of motion

$$\frac{\partial \vec{v_1}}{\partial t} + (\vec{v_0}.\vec{\nabla})\vec{v_1} + (\vec{v_1}.\vec{\nabla})\vec{v_0} = -\vec{\nabla}(h_1 + \Phi_1)$$

- $(\vec{v_0}.\nabla)\vec{v_1} = R\Omega_0 \frac{1}{R}\frac{\partial}{\partial \phi}\left(v_{\phi 1}\vec{u_\phi} + v_{R1}\vec{u_R}\right) = -\Omega_0 v_{\phi 1}\vec{u_R} + \Omega_0 v_{R1}\vec{u_\phi}$
- $(\vec{v_1}.\nabla)\vec{v_0} = \left(v_{\phi 1}\frac{1}{R}\frac{\partial}{\partial \phi} + v_{R1}\frac{\partial}{\partial R}\right)\left(R\Omega_0\vec{u_\phi}\right) = -v_{\phi 1}\Omega_0\vec{u_R} + v_{R1}\Omega_0\vec{u_\phi}$

So:

$$\boxed{\frac{\partial v_{R1}}{\partial t} - 2\Omega_0 v_{\phi 1} = -\frac{\partial}{\partial R}(h_1 + \Phi_1)}$$

$$\boxed{\frac{\partial v_{\phi 1}}{\partial t} + 2\Omega_0 v_{R1} = 0}$$

Linearized mass conservation

$$\frac{\partial \Sigma_1}{\partial t} + \vec{\nabla}.(\Sigma_0 \vec{v_1}) + \vec{\nabla}.(\Sigma_1 \vec{v_0}) = 0$$

- $\vec{\nabla}.(\Sigma_0 \vec{v_1}) = \Sigma_0 \frac{1}{R} \frac{\partial}{\partial R} (R v_{R1}) = \Sigma_0 \left(\frac{\partial v_{R1}}{\partial R} + \frac{v_{R1}}{R} \right)$
- $\vec{\nabla}.(\Sigma_1 \vec{v_0}) = 0$

So:

$$\boxed{\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \left(\frac{\partial v_{R1}}{\partial R} + \frac{v_{R1}}{R} \right) = 0}$$

Small perturbations and Poisson equation

▶ **Normal modes:**

$\Sigma_1(R,t) = \Sigma_a e^{i(\omega t - kR)}$
$\vec{v_1}(R,t) = \vec{v_a} e^{i(\omega t - kR)}$
$\Phi_1(R, z=0, t) = \Phi_a e^{i(\omega t - kR)}$ in the plane of the disk

▶ **Local WKB assumption:** We assume that the perturbation scales are small compared to those of the system: $kR \gg 1$ (for Wentzel - Kramers - Brillouin, as in quantum physics).

▶ **Full expression of Φ_1:**

- $\nabla^2 \Phi_1 = 0$ at $z \neq 0$
- $\Phi_1 = \Phi_a e^{i(\omega t - kR)}$ at $z = 0$

Away from the disk,

$$\nabla^2 \Phi_1 = (-i\frac{k}{r} - k^2)\Phi_1 + \frac{\partial^2 \Phi_1}{\partial z^2} = 0$$

WKB approximation $k/r \ll k^2$ so

$$\frac{\partial^2 \Phi_1}{\partial z^2} - k^2 \Phi_1 = 0$$

And

$$\boxed{\Phi_1(R,z,t) = \Phi_a e^{i(\omega t - kR) - |kz|}}$$

▶ **Poisson equation:**

$$\nabla^2 \Phi_1 = 4\pi G \Sigma_1 \delta(z)$$

We integrate Poisson equation from $z = -\epsilon$ to $+\epsilon$ and take the limit $\epsilon \to 0$, taking advantage of the continuity of Φ_1 along R:

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \nabla^2 \Phi_1 dz &= \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \frac{\partial^2 \Phi_1}{\partial z^2} dz \\
&= \lim_{\epsilon \to 0} \left[\frac{\partial \Phi_1}{\partial z} \right]_{-\epsilon}^{+\epsilon} \\
&= -2|k|\Phi_a e^{i(\omega t - kR)} \\
4\pi G \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \Sigma_1 \delta(z) dz &= 4\pi G \Sigma_a e^{i(\omega t - kR)}
\end{aligned}$$

$$\boxed{-2|k|\Phi_a = 4\pi G \Sigma_a}$$

Dispersion relation

$$\begin{cases} \dfrac{\partial v_{R1}}{\partial t} - 2\Omega_0 v_{\phi 1} = -\dfrac{\partial}{\partial R}(h_1 + \Phi_1) \\ \dfrac{\partial v_{\phi 1}}{\partial t} + 2\Omega_0 v_{R1} = 0 \\ \dfrac{\partial \Sigma_1}{\partial t} + \Sigma_0 \left(\dfrac{\partial v_{R1}}{\partial R} + \dfrac{v_{R1}}{R} \right) = 0 \\ -2|k|\Phi_1 = 4\pi G \Sigma_1 \\ h_1 = c_0^2 \dfrac{\Sigma_1}{\Sigma_0} \end{cases}$$

$$\begin{cases} i\omega v_{Ra} - 2\Omega_0 v_{\phi a} = ik(h_a + \Phi_a) \\ i\omega v_{\phi a} + 2\Omega_0 v_{Ra} = 0 \\ i\omega \partial \Sigma_a + \Sigma_0 \left(-ik + \dfrac{1}{R} \right) v_{Ra} = 0 \\ -2|k|\Phi_a = 4\pi G \Sigma_a \\ h_a = c_0^2 \dfrac{\Sigma_a}{\Sigma_0} \end{cases}$$

From which we can derive the dispersion relation $\boxed{\omega^2 - 4\Omega_0^2 + 2\pi G|k|\Sigma_0 - k^2 c_0^2 = 0}$

This dispersion relation can be rewritten as $\omega^2 = \left(c_0 k - \dfrac{\pi G \Sigma_0}{c_0}\right)^2 - \left(\dfrac{\pi G \Sigma_0}{c_0}\right)^2 + 4\Omega_0^2$

And introducing $\boxed{Q = \dfrac{2 c_0 \Omega_0}{\pi G \Sigma_0}}$, $\omega^2 = \left(c_0 k - \dfrac{\pi G \Sigma_0}{c_0}\right)^2 - \dfrac{1}{Q^2}(1 - Q^2)$

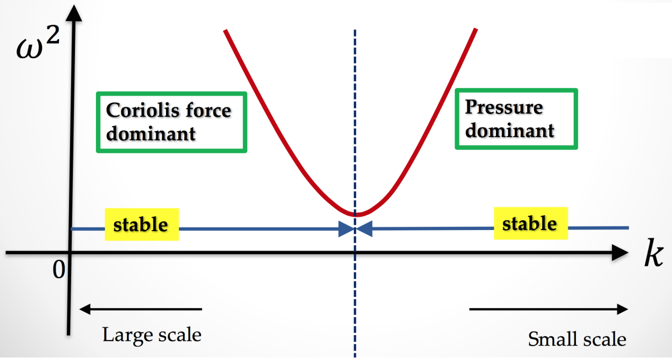
- If $Q \geq 1$, ω^2 always positive: the system is stable for all k.
- If $Q < 1$, ω^2 is negative for some values of k: the system is unstable.

Q is the **Toomre parameter.**

Dispersion relation

$$\omega^2 - 4\Omega_0^2 + 2\pi G|k|\Sigma_0 - k^2 c_0^2 = 0$$
$$\omega^2 = \left(c_0 k - \frac{\pi G \Sigma_0}{c_0}\right)^2 - \frac{1}{Q^2}(1 - Q^2)$$

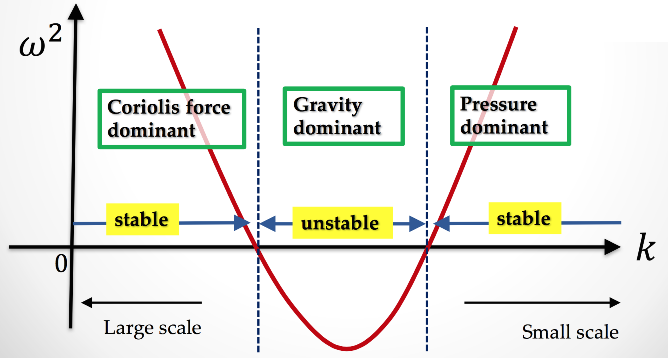
$\boldsymbol{Q \geq 1}$:



Dispersion relation

$$\omega^2 - 4\Omega_0^2 + 2\pi G|k|\Sigma_0 - k^2 c_0^2 = 0$$
$$\omega^2 = \left(c_0 k - \frac{\pi G \Sigma_0}{c_0}\right)^2 - \frac{1}{Q^2}(1 - Q^2)$$

$Q < 1$:



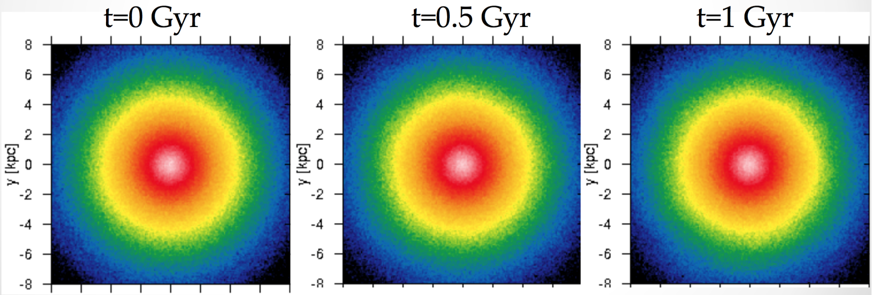
Limits of this calculation

- We assumed solid rotation: in the more general case, $2\Omega_0$ should be replaced by the epicyclic frequency $\kappa = \sqrt{r\frac{d\Omega_0^2}{dr} + 4\Omega_0^2}$
- We assumed a fluid and thus did not take stars into account
- We neglected magnetic fields
- Turbulence could be included inside the pressure term (taking a turbulent velocity instead of the sound speed), but its effects might be more complex: shock waves induce local compression that can trigger gravitational collapse. Compressive and solenoidal modes.
- Accretion and mergers affect disk stability

Effects on the disk

A numerical N-body simulation of a disk with different Toomre Q parameters (Shigeki Inoue):

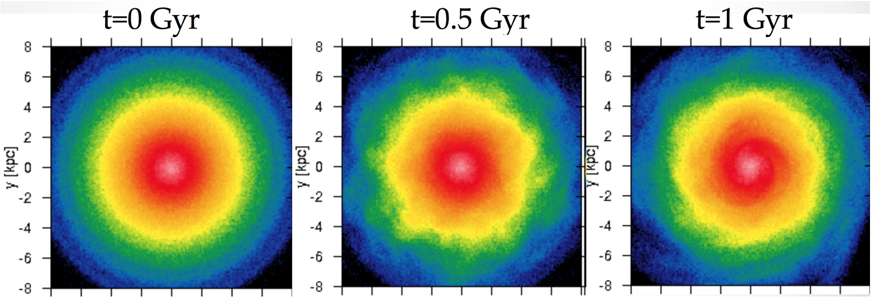
$Q = 2$:



Effects on the disk

A numerical N-body simulation of a disk with different Toomre Q parameters (Shigeki Inoue):

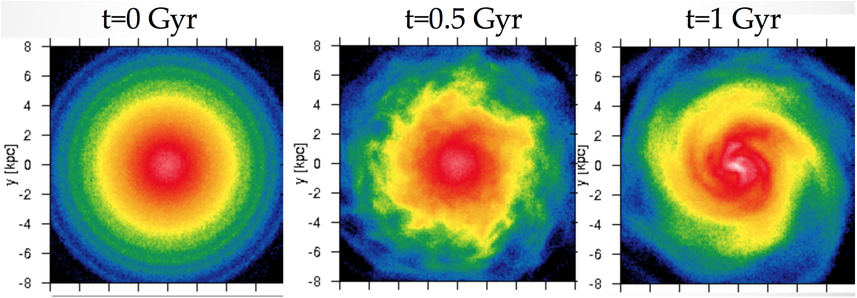
$Q = 1$:



Effects on the disk

A numerical N-body simulation of a disk with different Toomre Q parameters (Shigeki Inoue):

$Q = 0.7$:



Effects on the disk

A numerical N-body simulation of a disk (Anaelle Halle):

– spiral arms
– development of a bar
– star-forming clumps

Outlines

Summary

- **The Jeans instability:**

– Spherical systems (giant molecular clouds)
– Gravity vs. pressure
– Criterion: perturbations of size larger than the Jeans length $\lambda_{\rm Jeans} = \sqrt{\pi c_0^2 / G\rho_0}$ are unstable.

- **The Toomre instability:**

– Disks (galactic or proto-stellar)
– Gravity vs. pressure and rotation
– Criterion: when the Toomre parameter $Q = c_0\kappa/\pi G\Sigma_0$ is below unity, the disk is unstable.