COSMOLOGY – PROBLEM SET No. 1

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1. The Steady-State Model

The steady-state model assumes that the mean density in the universe and the Hubble constant are both constant in time, $\rho(t) = \rho_0$ and $H(t) = H_0$. The Hubble expansion is compensated by a continuous creation of matter everywhere.

a. In a sphere of radius R containing mass M, compute the rate of mass creation $\dot{M}$ needed to compensate for the expansion, and then the relative change of mass $\Delta M/M$ during time interval Δt.

b. Given that $\rho_0 \sim 10^{-29} g\, cm^{-3}$ and $H_0 \sim 70\,\mathrm{km\,s^{-1}Mpc^{-1}}$, compute the total mass creation per year inside a room ($10^3 cm$), and in the whole solar system ($\sim 10^{13} cm$). Is this detectable?

c. Draw a diagram showing radius as a function of time for comoving shells of matter, for the steady-state cosmology [$H(t)$ =const.] in comparison with the standard big-bang cosmology [ignore gravity, $v(t)$ =const.].

2. Homogeneity, Isotropy and the Hubble Expansion

a. Can the universe be inhomogeneous and still be globally isotropic?

b. Assuming the universe is homogeneous, does this imply it is also globally isotropic?

c. The Hubble law is $\vec{v} = H_0 \vec{r}$. Show that (in a flat universe):

   i. It is isotropic about any comoving observer.

   ii. It is the only velocity field permissible by the assumption of spatial homogeneity (at any given time) and isotropy (at any given spatial point). This is called "the cosmological principle".

3. Areas and Volumes in the Robertson-Walker Metric

Assuming homogeneity and flat space locally, the line element on a 3-D surface of constant t, using comoving spherical coordinates u, θ and ϕ, is

$$d\ell^2 = a^2(t)\,[du^2 + S_k^2(u)\,d\gamma^2],$$

where $a(t)$ is the universal expansion factor, u is the comoving radial coordinate ($u = r/a$), $d\gamma$ is the usual area element on a unit sphere,

$$d\gamma^2 = d\theta^2 + \sin^2\theta\,d\phi^2,$$

and

$$S_k(u) = \begin{cases} \sin(u) & k = 1 \\ u & k = 0 \\ \sinh(u) & k = -1 \end{cases}.$$

a. Calculate in a closed universe ($k = 1$) the area A of a sphere of comoving radius u.

b. How does A vary with growing u? Does it increase for all values of u? Try to visualize by drawing a 2-D sphere for the universe (ϕ =const.) and considering the circumference of a circle on it (u =const.).

c. Calculate the total volume of a closed universe as a function of $a(t)$.

d. Repeat a-c for an open universe ($k = -1$).

Try to visualize the three geometries of the Robertson-Walker metric. Guidance is provided in box 27.2 (page 723) in Gravitation by Misner, Thorne and Willer.

4. Representations of the Robertson-Walker metric

Consider the *isotropic* form of the line element (3-D space),

$$d\ell^2 = (dx^2 + dy^2 + dz^2)/u(R),$$

where $u(R) = [1 + 0.25 \cdot KR^2]^2, R^2 = x^2 + y^2 + z^2$.

a. By transforming to different coordinate system (r, θ, ϕ), write this metric as:

$$d\ell^2 = dr^2/[1 - Kr^2] + r^2(d\theta^2 + sin^2\theta d\phi^2).$$

b. Find a further change of radial coordinates to bring it to the standard RW form ($K > 0$):

$$d\ell^2 = K^{-1} \cdot [d\chi^2 + sin^2(\chi)(d\theta^2 + sin^2\theta d\phi^2)].$$

c. Define the conformal time parameter by $\eta = c \int_0^t \frac{dt'}{a(t')}$. Find the form of the RW metric when t is replaced with η.

5. Expansion with constant rate

Assume the time dependence of RW expansion factor is

$$a(t) = t,$$

and the universe is open ($k = -1$). This expansion is the solution of Friedmann equation for negligible matter & radiation density (with $\Lambda = 0$).

a. Write the RW line element. Preform the coordinate change:

$$r = t \sinh(\chi),$$

$$\tau = t \cosh(\chi),$$

and write the new line element in terms of r, τ.

b. What kind of metric did you get? Does the 3-space geometry is curved?

6. The Kasner metric

Consider the Kasner metric:

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2,$$

where p_i are constants satisfying:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$$

a. Is Kasner metric isotropic? homogeneous?

b. Compute its volume element.

c. Describe the velocity field seen by a comoving observer.