Instability in astronomical objects: spheroids and discs

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Gravitational instabilities

- Jeans instability
  - 3-dimensional instability
  - gravity vs pressure
  - mainly for spherical systems
    - gas clouds, spherical (elliptical) galaxies, clusters of galaxies, etc…

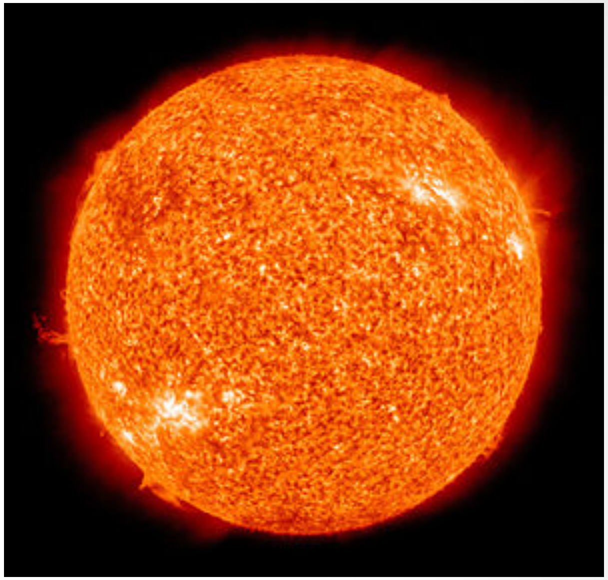
- Toomre instability
  - 2-dimensional instability
  - gravity vs pressure+Coriolis force
  - for disc systems
    - accretion discs around stars, disc galaxies

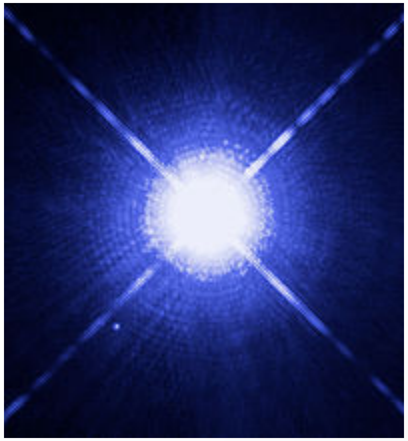
Jeans instability

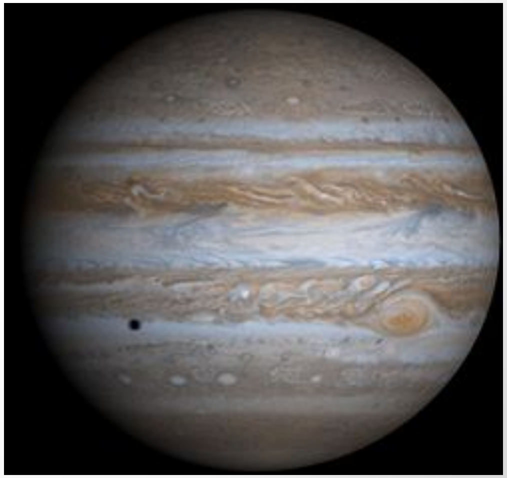
- Gas clouds
- Can they exist forever?
  - Can they sustain their shapes?
- Probably, no.
  - Gas clouds would collapse into stars.

Jeans instability

- Stars (and a planet)
  - They are also made of gas.
- Can they exist forever?
  - Can they sustain their shapes?
- Yes.
- They would not collapse.
  - as long as atomic nuclei are burning



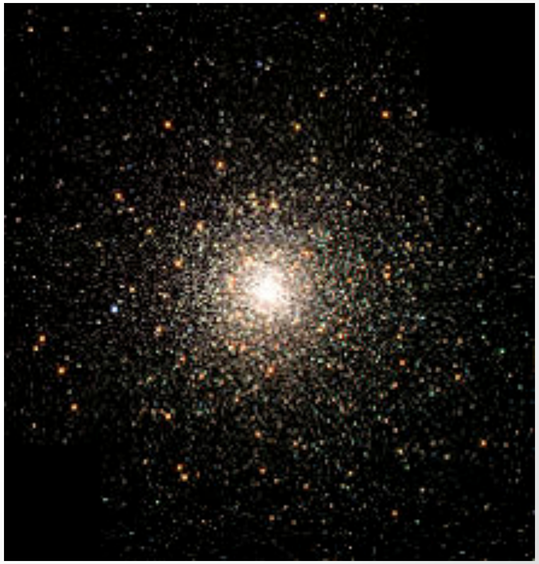




Jeans instability

- Elliptical galaxies
- Globular clusters

- Can they exist forever?
  - Can they sustain their shapes?
- Yes.
  - Indeed, they are quite old.
  - ~10 Gyr



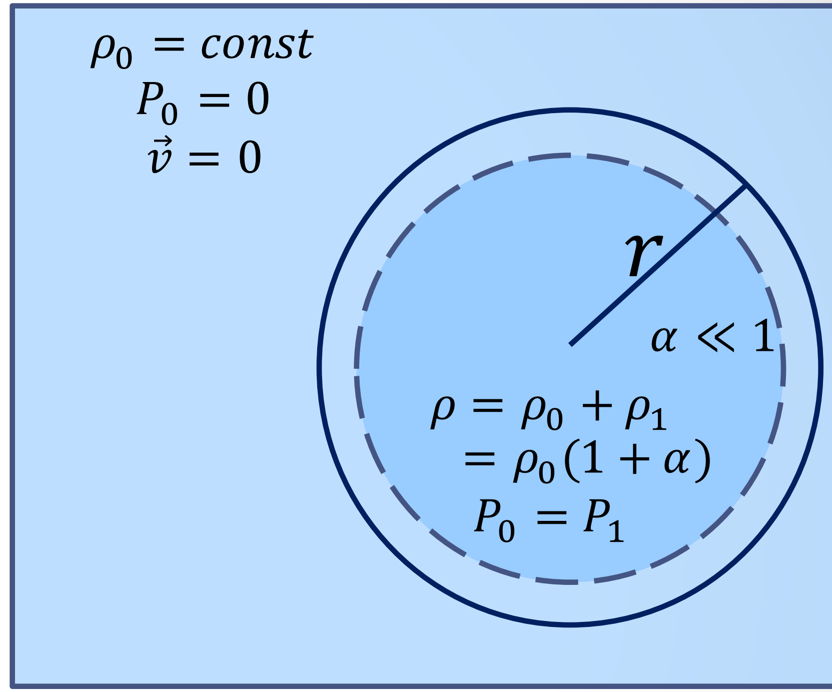
What determines the stability?

- The balance between gravity and pressure
  - Gravity
    - contracting force
  - pressure (gas), velocity dispersion (stars)
    - expanding force

  - If gravity exceeds pressure, the systems will collapse.
    - forever (black holes)
    - stopped by something (star formation, etc.)
  - If pressure exceeds gravity, the systems will expand until the pressure becomes weaker than the gravity.

Jeans instability

- Basic physics



Gravity

$$F_{G1} = \nabla \Phi_1 = \frac{\frac{4}{3}\pi r^3 G \rho_1}{r^2}$$

$$\sim \pi G \rho_1 r = \pi \alpha G \rho_0 r$$

Pressure

$$P_1 = \left({}^{dP}\!/_{d\rho}\right)_0 d\rho = C_s{}^2 \rho_1 = \alpha C_s{}^2 \rho_0 \quad (C_s\text{: sound speed})$$

$$F_{P1} = {}^{\nabla P_1}\!/_{\rho_0} \sim \frac{P_1}{r\rho_0} = \frac{\alpha C_s{}^2}{r} \qquad \left(\leftarrow \nabla \sim \frac{1}{r}\right)$$

- For instability,

$$F_{G1} > F_{P1}$$

$$\alpha G \rho_0 r > \frac{\alpha {C_s}^2}{r}$$

$$r > \frac{C_s}{\sqrt{G\rho_0}}$$

A large scale tends to be unstable, but a small scale is supported by pressure.

Dynamical time $t_{dyn} \equiv \frac{1}{\sqrt{G\rho_0}}$

$$\frac{r}{C_s} > t_{dyn}$$

If sound wave propagates faster than perturbation, the system is stable.

Jeans instability for fluid

- More accurate version

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \quad \text{------①}$$

$$\frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla}(h_1 + \Phi_1) \quad \text{------②}$$

$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \quad \text{------③}$$

$$h_1 = \frac{dP_1}{d\rho}\frac{d\rho}{\rho_0} = {C_s}^2 \frac{\rho_1}{\rho_0} \quad \text{------④}$$

h : specific enthalpy

From time-derivative of ① and divergence of ②

$$\frac{\partial(\vec{\nabla} \cdot \vec{v}_1)}{\partial t} = -\nabla^2(h_1 + \Phi_1) \quad \text{------⑥}$$

Using all above equations,

$$\frac{\partial^2 \rho_1}{\partial t^2} - {C_s}^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0 \quad \text{------⑦}$$

Assuming

$$\rho_1(\vec{x}, t) = C \exp\left[i\left(\vec{k} \cdot \vec{x} - \omega t\right)\right] \quad \text{------⑧}$$

If $\omega^2 > 0$, stable (sinusoidal wave).

If $\omega^2 < 0$, unstable (exponential growth).

Substitute ⑧ into ⑦,

$$\omega^2 = {C_s}^2 k^2 - 4\pi G \rho_0 \quad : \quad \text{dispersion relation}$$

- When $\omega = 0$ (the boundary case),

$$k_J{}^2 = \frac{4\pi G \rho_0}{C_s{}^2}$$

$$\lambda_J{}^2 = \frac{\pi C_s{}^2}{G \rho_0} \quad : \text{ Jeans wavelength}$$

$$M_J = \frac{4\pi}{3}\rho_0\left(\frac{\lambda_J}{2}\right)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi C_s{}^2}{G\rho_0}\right)^{\frac{3}{2}} \quad : \text{ Jeans mass}$$

Uniform fluid will collapse into gas clouds of M_J.

Jeans instability for stars

- $f(\vec{x}, \vec{v}, t)$: distribution function
  time-dependent 6-D density of stars

$$f = f_0 + f_1$$

$$\frac{Df_1}{Dt} = \frac{\partial f_1}{\partial t} + \vec{v}\frac{\partial f_1}{\partial \vec{x}} + \nabla\Phi_1\frac{\partial f_0}{\partial \vec{v}} = 0 \quad \text{------⑨}$$

$$\nabla^2\Phi_1 = 4\pi G\rho_1 = 4\pi G\int f_1\, d\vec{v} \quad \text{------⑩}$$

Assuming

$$f_1(\vec{x}, \vec{v}, t) = f_a(\vec{v})\exp\left[i\left(\vec{k}\cdot\vec{x} - \omega t\right)\right]$$

$$\Phi_1(\vec{x}, t) = \Phi_a\exp\left[i\left(\vec{k}\cdot\vec{x} - \omega t\right)\right]$$

Substituting them into ⑨ and ⑩

$$\left(\vec{k}\cdot\vec{x}-\omega\right)f_a-\Phi_a\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}=0$$

$$-k^2\Phi_a=4\pi G\int f_a\,d\vec{v}$$

Combining them,

$$1+\frac{4\pi G}{k^2}\int\frac{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}}{\vec{k}\cdot\vec{v}-\omega}d\vec{v}=0 \quad \text{------⑪}$$

Assuming f_0 to be a Maxwellian DF,

$$f_0(\vec{v})=\frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\exp\left(-\frac{v^2}{2\sigma^2}\right) \quad \text{------⑫}$$

σ : velocity dispersion

Maxwell distribution : isotropic and Gaussian

Substitute ⑫ into ⑪, and consider along x-axis

$$1-\frac{2\sqrt{2\pi}G\rho_0}{k_x\sigma^3}\int_{-\infty}^{\infty}\frac{v_x\exp\left(-\frac{{v_x}^2}{2\sigma^2}\right)}{k_xv_x-\omega}dv_x=0 \qquad \text{------⑬}$$

When $\omega = 0$ (the boundary case), ⑬ becomes

$$1-\frac{4\pi G\rho_0}{k_J\sigma^2}=0$$

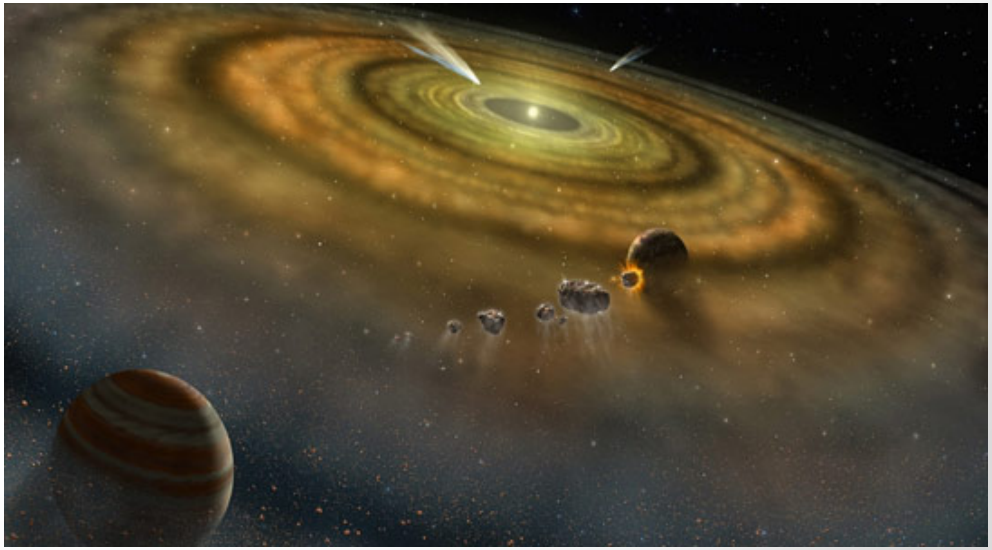
$$ {k_J}^2=\frac{4\pi G\rho_0}{\sigma^2}, \qquad {\lambda_J}^2=\frac{\pi\sigma^2}{G\rho_0}$$

Replacing σ with C_s, k_J is the same as in the fluid case.

$k_{J,gas} = k_{J,star}$

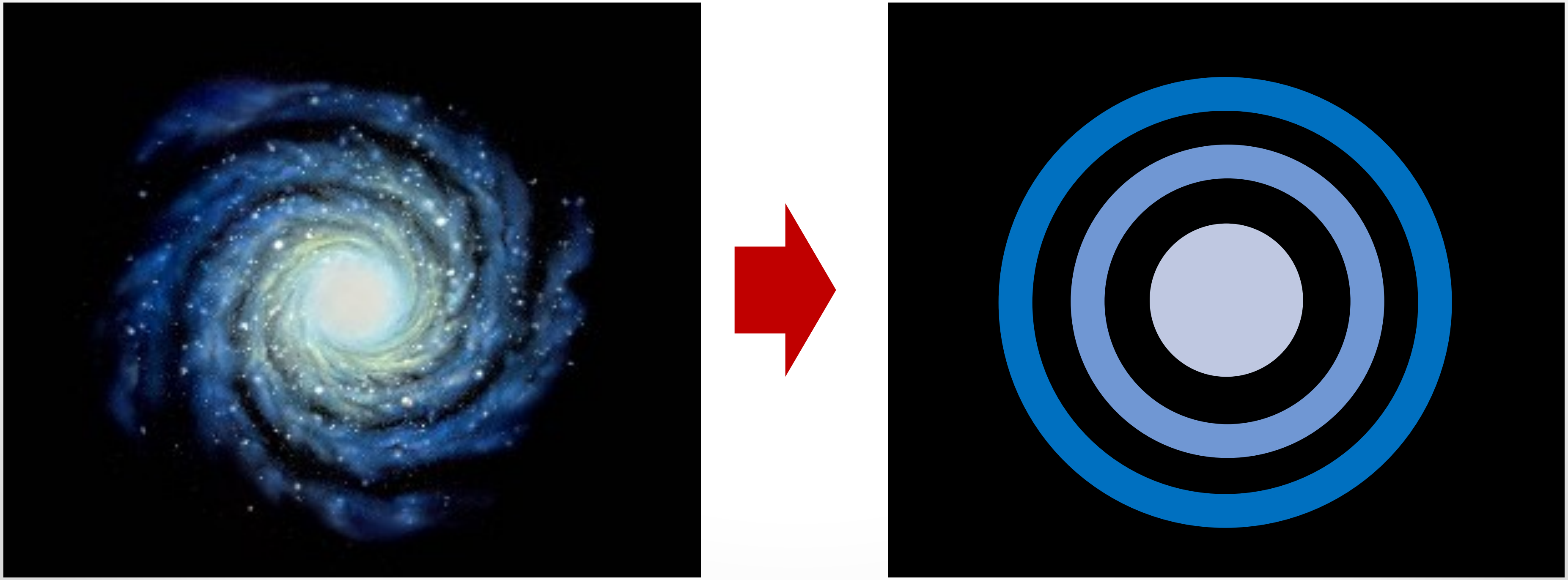
Toomre instability

- 2-dimensional instability
  - Galactic discs
  - Proto-planetary discs
- Toomre instability may be related to
  - Star cluster / gas cloud formation
  - Planet formation



Toomre instability

- A disc may be unstable against perturbation by “spiral-arm-like” structures.
- They can be approximated to be “ripple waves”.
    - Gravity vs Pressure+Coriolis force



Toomre instability

- What would happen if Toomre unstable?

Bar formation?
Spiral arms?
Stellar clump formation?

Toomre instability

- 2-D disc $\quad\quad \rho \Rightarrow \Sigma, \quad r \Rightarrow R$

$$\frac{\partial \Sigma}{\partial t} + \vec{\nabla} \cdot (\Sigma \vec{v}) = 0 \qquad \text{------①}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}(h + \Phi) \qquad \text{------②}$$

$$\nabla^2 \Phi = 4\pi G \Sigma \delta(z) \qquad \text{------③}$$

Axisymmetric perturbation

$$\frac{\partial}{\partial \phi} = 0, \quad kR \gg 1$$

(Terms of $\frac{1}{R}$ are negligible compared with terms of k)

Linear perturbation theory

$$\Sigma = \Sigma_0 + \varepsilon\Sigma_1 , \quad v_R = \varepsilon v_{R1} \quad (v_{R0} = 0),$$

$$v_\phi = v_{\phi 0} + \varepsilon v_{\phi 1} , \quad h = h_0 + \varepsilon h_1 ,$$

$$\Phi = \Phi_0 + \varepsilon\Phi_1$$

Terms of $\varepsilon^0, \varepsilon^2, \varepsilon^3 \cdots$ are ignored, only ε is considered.

i.e.
$$AB = A_0 B_0 + \varepsilon(A_0 B_1 + A_1 B_0) + \varepsilon^2 A_1 B_1$$
$$\sim \varepsilon(A_0 B_1 + A_1 B_0)$$

By linearizing, ① becomes

$$\frac{\partial \Sigma_1}{\partial t} = \Sigma_0 \left(\frac{\partial v_{R1}}{\partial R} + \frac{v_{R1}}{R} \right) \quad \text{------④}$$

Assuming

$$\Sigma_1 = \Sigma_a \exp[i(kR - \omega t)]$$

$$v_{R1} = v_{Ra} \exp[i(kR - \omega t)],$$

④ becomes

$$-i\omega\Sigma_a + ik\Sigma_0 v_{Ra} = 0 \quad \text{------⑤}$$

($\frac{v_{R1}}{R}$ is negligible since $\frac{1}{R} \ll k$)

By linearizing, ② becomes

$$\frac{\partial v_{R1}}{\partial t} - 2\Omega v_{\phi 1} = -\frac{\partial}{\partial R}(\Phi_1 + h_1) \quad \text{------⑥}$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \frac{\kappa^2}{2\Omega} v_{R1} = 0 \quad \text{------⑦}$$

$\Omega \equiv \frac{v_{\phi 0}}{R}$: angular velocity

$\kappa^2 \equiv 2\frac{v_{\phi 0}}{R}\left(\frac{v_{\phi 0}}{R} + \frac{\partial v_{\phi 0}}{\partial R}\right)$: epicyclic frequency

$$h_1 = \frac{dP}{\Sigma} = \left(\frac{dP}{d\Sigma}\right)_0 \frac{d\Sigma}{\Sigma_0} = {C_s}^2 \frac{\Sigma_1}{\Sigma_0} \quad \text{------⑧}$$

Assuming

$$v_{\phi 1} = v_{\phi a} \exp[i(kR - \omega t)]$$

$$\Phi_1 = \Phi_a \exp[i(kR - \omega t)] \qquad \text{(at } z = 0\text{)}$$

$$h_1 = h_a \exp[i(kR - \omega t)],$$

⑥ and ⑦ become, with ⑧,

$$-i\omega v_{Ra} - 2\Omega v_{\phi a} = -ik\left(\Phi_a + {C_s}^2 \frac{\Sigma_a}{\Sigma_0}\right)$$

$$-i\omega v_{\phi a} + \frac{\kappa^2}{2\Omega} v_{Ra} = 0$$

The above two are combined into

$$v_{Ra} = -\frac{\omega k}{\kappa^2 - \omega^2}\left(\Phi_a + {C_s}^2 \frac{\Sigma_a}{\Sigma_0}\right) \qquad \text{------⑨}$$

By linearizing, ③ becomes

$$\nabla^2 \Phi_1 = 4\pi G \Sigma_1 \delta(z) \qquad \text{------⑩}$$

⑩ has to be

$$\nabla^2 \Phi_1 = 0 \qquad (\text{at } z \neq 0)$$

$$\Phi_1 = \Phi_a \exp[i(kR - \omega t)] \qquad (\text{at } z = 0)$$

The only solution is

$$\Phi_1 = \Phi_a \exp[i(kR - \omega t) - k|z|]$$

Integrate ⑩ over z = $[-\epsilon, \epsilon]$,

$$\int_{-\epsilon}^{\epsilon} \left(\frac{\partial^2 \Phi_1}{\partial R^2} + \frac{\partial^2 \Phi_1}{\partial z^2} \right) dz = 2 \left. \frac{\partial \Phi_1}{\partial z} \right|_{z=\epsilon} = 4\pi G \Sigma_1 \int_{-\epsilon}^{\epsilon} \delta(z)\, dz$$

$$\lim_{\epsilon \to 0} 2 \left. \frac{\partial \Phi_1}{\partial z} \right|_{z=\epsilon} = 2 \left. \frac{\partial \Phi_1}{\partial z} \right|_{z=0} = 4\pi G \Sigma_1$$

Then,

$$-k\Phi_a = 2\pi G\Sigma_a$$

$$\Phi_a = -\frac{2\pi G}{k}\Sigma_a \qquad \text{------⑪}$$

Combine ⑤ and ⑨,

$$\frac{\omega}{k}\frac{\Sigma_a}{\Sigma_0} = -\frac{k\omega}{\kappa^2 - \omega^2}\left(\Phi_a + {C_s}^2\frac{\Sigma_a}{\Sigma_0}\right)$$

$$\Sigma_a = -\frac{k^2\Sigma_0}{\kappa^2 - \omega^2 + k^2{C_s}^2}\Phi_a$$

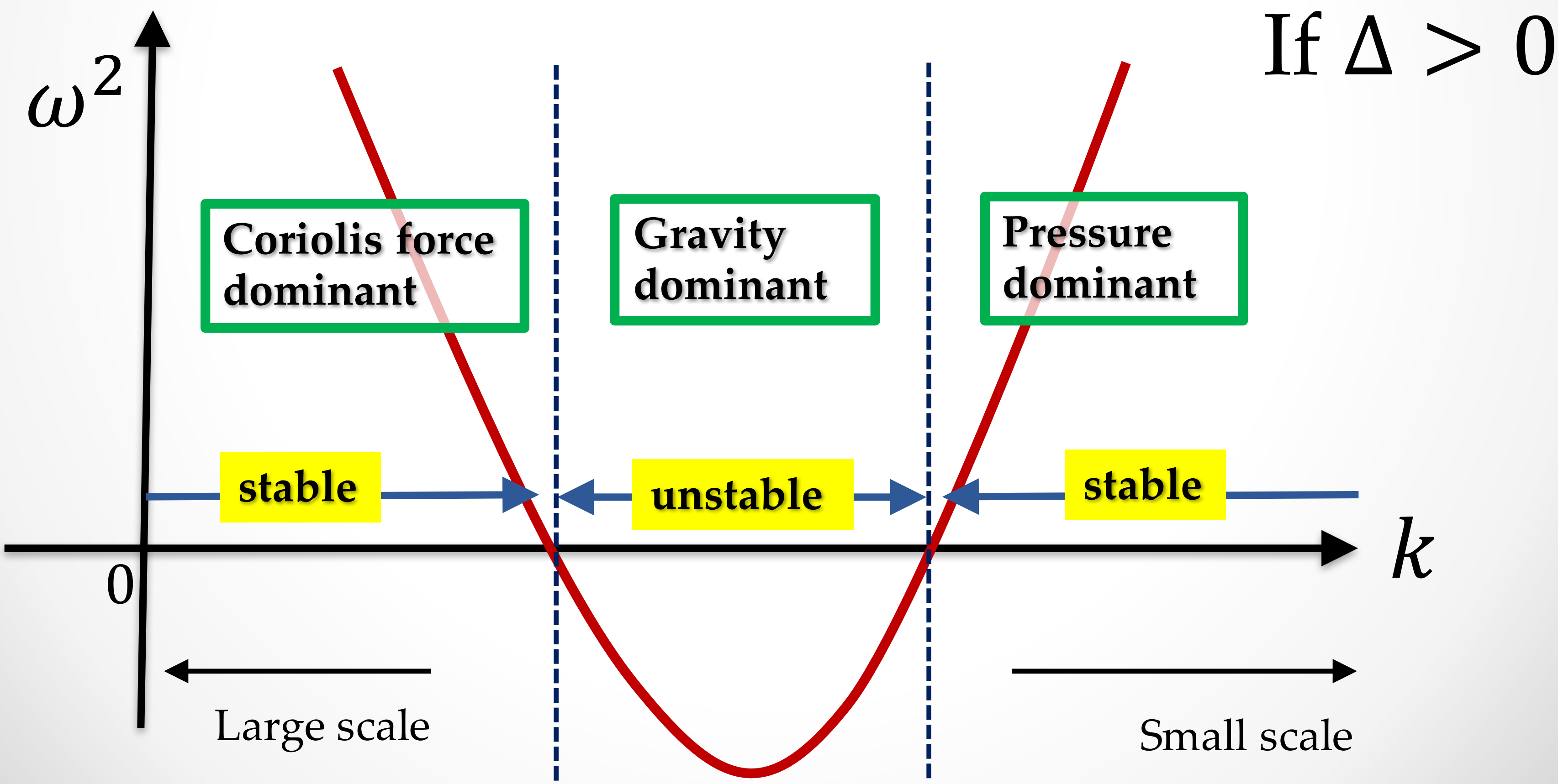
Substitute ⑪ into it,

$$2\pi Gk\frac{\Sigma_0}{\kappa^2 + k^2{C_s}^2 - \omega^2} = 1$$

Finally, we obtain a dispersion relation

$$\omega^2 = k^2 {C_s}^2 - 2\pi G k \Sigma_0 + \kappa^2$$

This is a quadratic equation of k.



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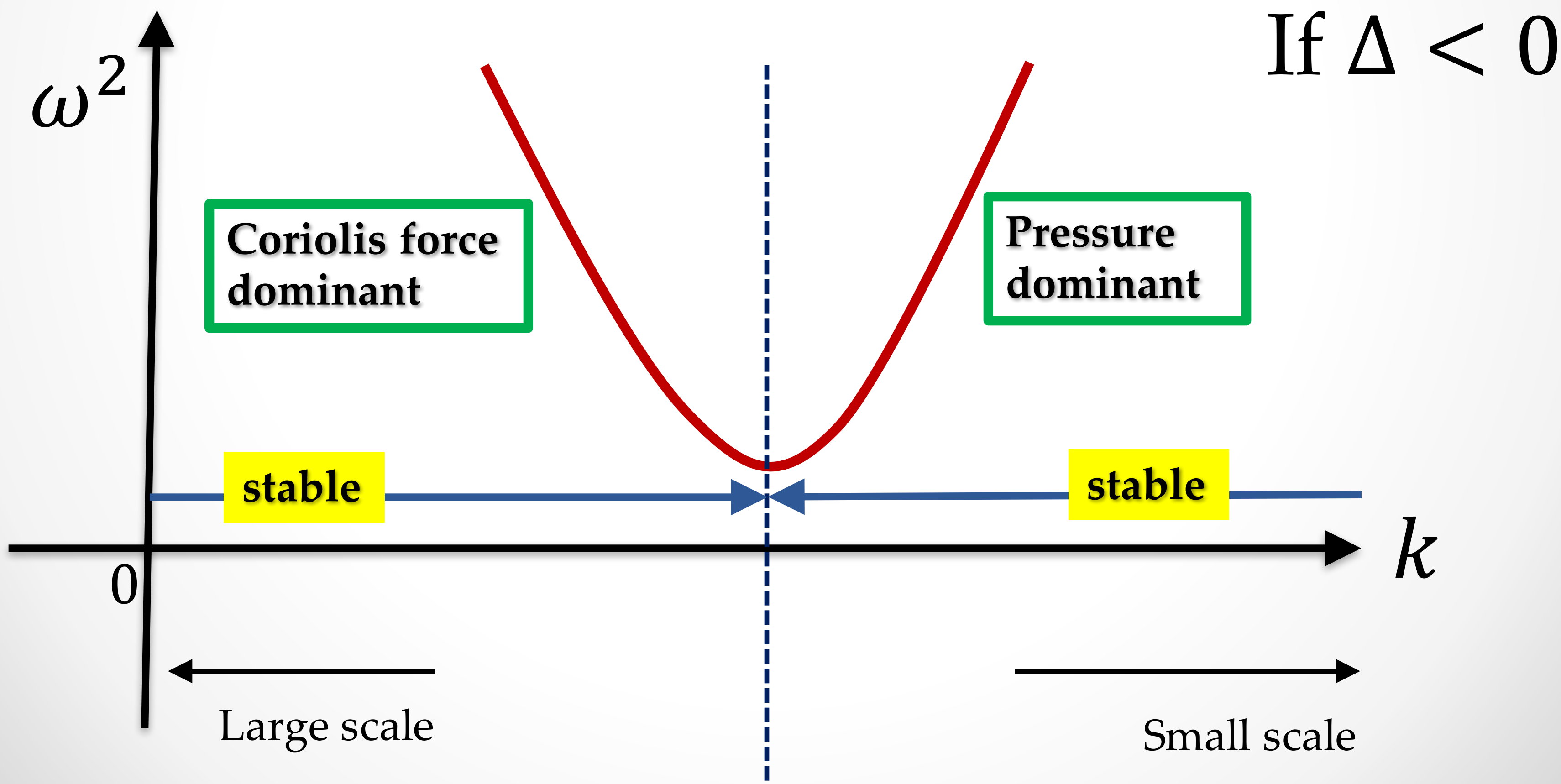


The discriminant of the dispersion relation is

$$Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0}$$ for razor-thin gas disc

This is called “Toomre’ s Q-parameter”.

If Q<1, disc is unstable,
if Q>1, disc is stable
for axisymmetric perturbation.

For stellar discs, $C_s \Rightarrow \sigma_R$.

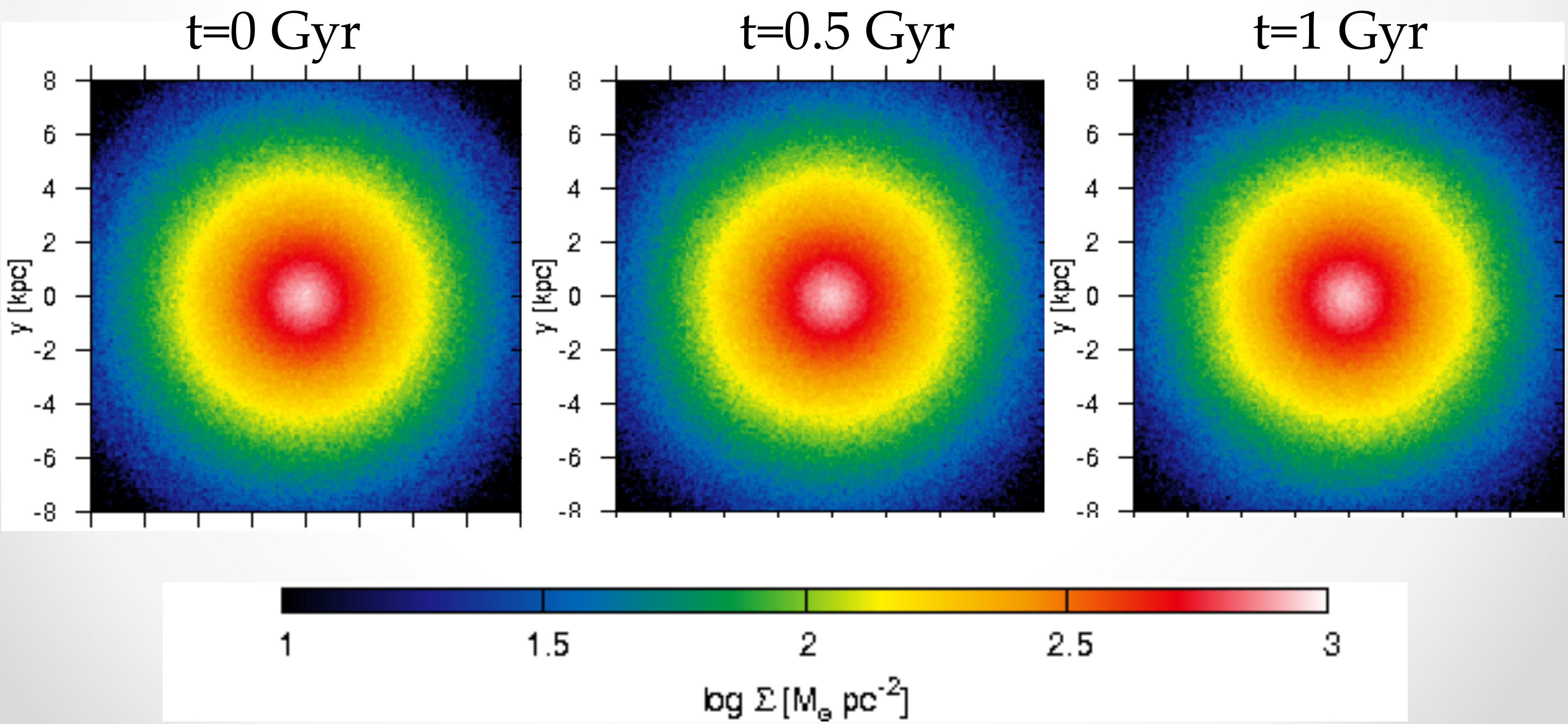
But, it is not exactly compatible. Then,

$$Q \equiv \frac{C_s \kappa}{3.36 G \Sigma_0}$$ for razor-thin stellar disc

For details, see Toomre (1964, ApJ, 139, 121) or "Galactic Dynamics" by Binney & Tremaine

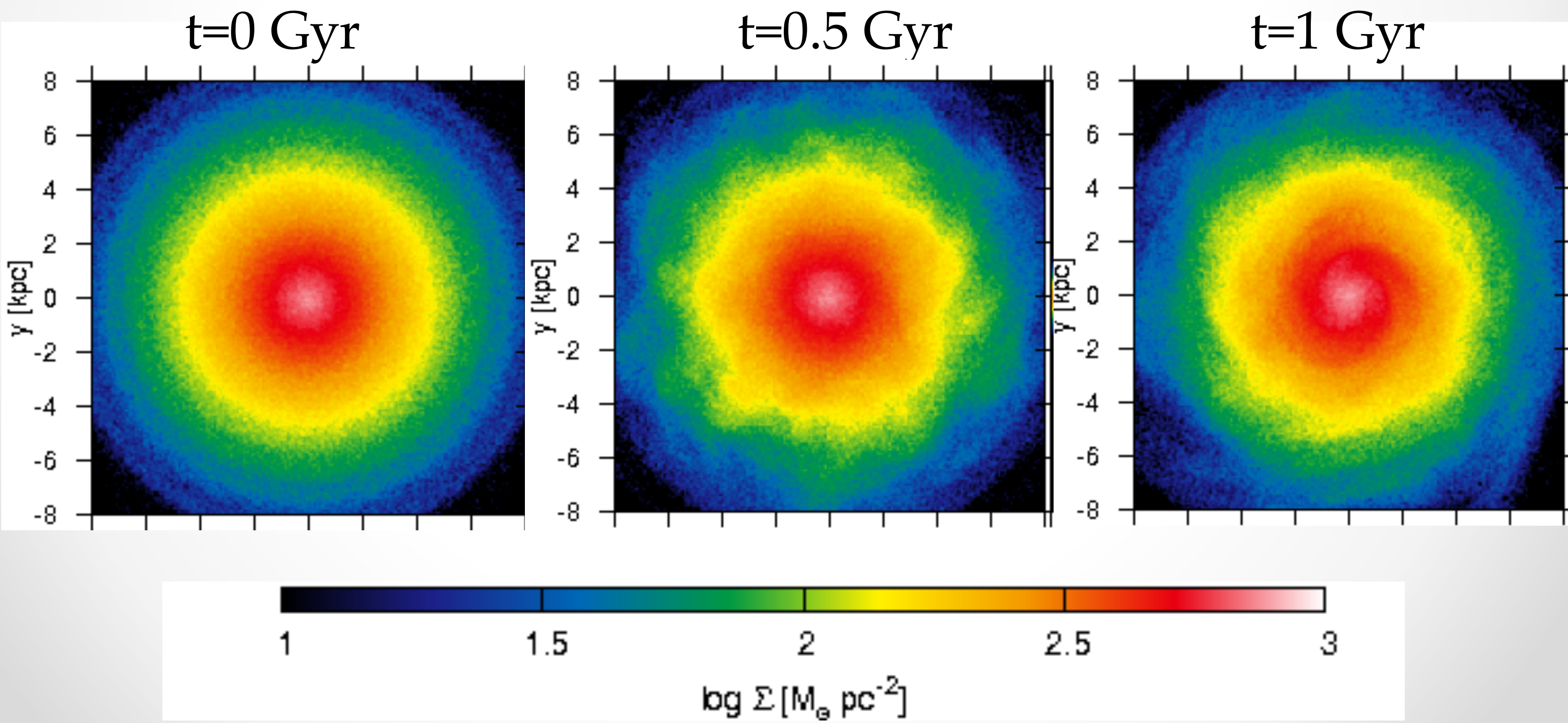
What happens if Toomre unstable

- N-body disc galaxy simulations
  - Q=2.0



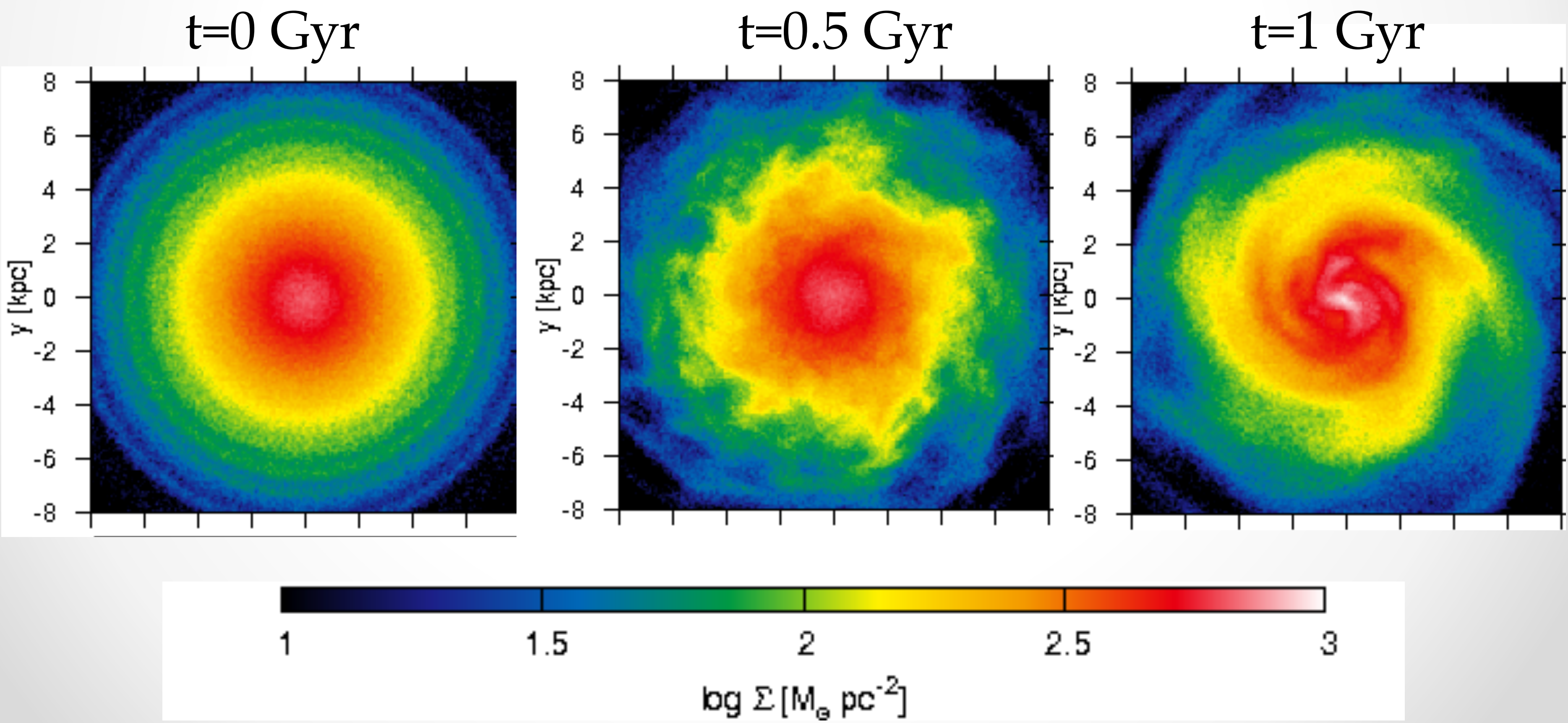
What happens if Toomre unstable

- N-body disc galaxy simulations
  - Q=1.0



What happens if Toomre unstable

- N-body disc galaxy simulations
  - Q=0.7



Summary

- Jeans instability
  - 3-D spatial instability
  - Gravity vs. pressure
  - $M_J = \frac{4\pi}{3}\rho_0\left(\frac{\lambda_J}{2}\right)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi {C_s}^2}{G\rho_0}\right)^{\frac{3}{2}}$: Jeans mass

- Toomre instability
  - 2-D disc instability
  - Gravity vs. Pressure + Coriolis force
  - $Q \equiv \frac{C_s\kappa}{\pi G\Sigma_0} > 1$ for a stable state in a thin gas disc