COSMOLOGY – PROBLEMS 1-10

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1. The Hubble Expansion

The Hubble law is $\boldsymbol{v} = H_0\boldsymbol{r}$. Show that (in a flat universe):

a. It is isotropic about any comoving observer.

b. It is the only velocity field permissible by the assumption of spatial homogeneity at any given time (called "the cosmological principle").

2. The Steady-State Model

The steady-state model assumes that the mean density in the universe and the Hubble constant are both constant in time, $\rho(t) = \rho_0$ and $H(t) = H_0$. The Hubble expansion is compensated by a continuous creation of matter everywhere.

a. In a sphere of radius R containing mass M, compute the rate of mass creation $\dot{M}$ needed to compensate for the expansion, and then the relative change of mass $\Delta M/M$ during time interval Δt.

b. Given that $\rho_0 \sim 10^{-29} g\, cm^{-3}$ and $H_0 \sim 70\,\mathrm{km\,s^{-1}Mpc^{-1}}$, compute the total mass creation per year inside a room ($10^3 cm$), and in the whole solar system ($\sim 10^{13} cm$). Is this detectable?

c. Draw a diagram showing radius as a function of time for comoving shells of matter, for the steady-state cosmology [$H(t)$ =const.] in comparison with the standard big-bang cosmology [ignore gravity, $v(t)$ =const.].

3. Areas and Volumes in the Robertson-Walker Metric

Assuming homogeneity and flat space locally, the line element on a 3-D surface of constant t, using comoving spherical coordinates u, θ and ϕ, is

$$d\ell^2 = a^2(t)\,[du^2 + S_k^2(u)\,d\gamma^2],$$

where $a(t)$ is the universal expansion factor, u is the comoving radial coordinate ($u = r/a$), $d\gamma$ is the usual area element on a unit sphere,

$$d\gamma^2 = d\theta^2 + \sin^2\theta\,d\phi^2,$$

and

$$S_k(u) = \begin{cases} \sin(u) & k = 1 \\ u & k = 0 \\ \sinh(u) & k = -1 \end{cases}.$$

a. Calculate in a closed universe ($k = 1$) the area A of a sphere of comoving radius u.

b. How does A vary with growing u? Does it increase for all values of u? Try to visualize by drawing a 2-D sphere for the universe ($\phi =$const.) and considering the circumference of a circle on it ($u =$const.).

c. Calculate the total volume of a closed universe as a function of $a(t)$.

d. Repeat a-c for an open universe ($k = -1$).

Try to visualize the three geometries of the Robertson-Walker metric. Guidance is provided in box 27.2 (page 723) in Gravitation by Misner, Thorne and Willer.

4. Newtonian Dynamics

Assume a 3-dimensional, Euclidean, homogeneous universe, with a spatially constant mass density $\rho(t)$ [today $\rho(t_0) = \rho_0$].

a. Consider a thin comoving spherical shell of matter of radius $r(t)$ [today $r(t_0) = r_0$]. Adopt the Birkhoff hypothesis that the gravitational acceleration at any point on the shell is determined by the matter interior to the shell. Write down the equation of motion for the shell [namely a relation between $\ddot{r}(t)$ and $r(t)$.

b. Use the fact that the homogeneity implies a universal expansion parameter $a(t)$ [namely, for every shell of comoving radius u the radius at time t is provided by $r(t) = a(t)u$] to obtain an equation of motion for $a(t)$ [independently of the specific shell considered].

c. Integrate to obtain an equation of energy conservation [involving a, $\dot{a}$, and a constant in time ϵ]. Show that for a given shell of comoving radius u, this constant is related to the total energy of the shell E via $\epsilon = 2E/u^2$.

d. Convince yourself that we can re-scale the distance units of u such that ϵ obtains one of the three possible values ± 1 or 0 for all the shells. Then ϵ is a constant parameter which characterizes the cosmological model to be one of three possibilities.

e. For which value of ϵ will the universe expand forever (an unbound universe), and for which will it eventually stop and turn-around to collapse into a Big Crunch (a bound universe)? What happens at $t \to \infty$ for $\epsilon = 0$?

f. Show that there is a **critical density**, ρ_c, for which the universe is marginally bound ($\epsilon = 0$). What is this critical density today given that the Hubble constant is $H_0 = 60\,\mathrm{km\,s^{-1}Mpc^{-1}}$ ($1\mathrm{pc} = 3 \times 10^{18}\mathrm{cm}$).

g. Try to relate the 3 dynamical models of $\epsilon = 0, \pm 1$ to the 3 geometrical models $k = 0, \pm 1$ of the Robertson-Walker metric.

5. Solving the Friedmann Equation, $\Lambda = 0$

Write the Friedmann equation in the matter-dominated era, for the case $\Lambda = 0$, as

$$\dot{a}^2 - \frac{2a^*}{a} = -k, \quad \text{where} \quad a^* \equiv \frac{4\pi}{3}\rho_{m0}\, a_0^3 = \text{const.} \tag{1}$$

a. What is the solution for $a(t)$ in the case $k = 0$?

b. What is the asymptotic solution at early times when a is small (for any k)?

c. For $k = -1$, what is the solution at late times, when a is large?

d. Solve it for the expansion factor $a(t)$ in the following steps.

- Define the *conformal time* η by

$$d\eta \equiv \frac{dt}{a(t)}. \tag{2}$$

  If we denote $d/d\eta$ by a prime, note that $\dot{a} = a'/a$, and replace in eq. (1) the variable t by the variable η.

- Now add to the game the time derivative of eq. (1) to obtain the differential equation

$$a'' + ka = a^*. \tag{3}$$

- Show that that the following are solutions of eq. (3) for the cases $k = \pm 1$:

$$a = ka^*[1 - C_k(\eta)] \quad \text{where} \quad C_k(\eta) \equiv \begin{cases} \cos\eta & \text{k=1} \\ \cosh\eta & \text{k=-1} \end{cases} \tag{4}$$

- In order to connect this to the cosmological time t, integrate eq. (2) to obtain

$$t = ka^*[\eta - S_k(\eta)] \quad \text{where} \quad S_k(\eta) \equiv \begin{cases} \sin\eta & \text{k=1} \\ \sinh\eta & \text{k=-1} \end{cases} \tag{5}$$

  Although this is not an explicit solution for $a(t)$, it is a useful implicit solution, involving the parameter η in the relation between a and t.

e. Show that in the limit of early times, $\eta \ll 1$, the solution converges to the $a(t)$ solution obtained for the limiting cases in (b) and (c).

f. Draw a qualitative plot of the solutions for $a(t)$ in the 3 cases of k. In particular note for a bound universe what is the conformal time at maximum expansion and at the big crunch, and what is a at maximum expansion.

g. In the radiation-dominates era, in the earlier universe, explain why the Friedmann equation takes the form

$$\dot{a}^2 - \frac{(a^\dagger)^2}{a^2} = -k, \quad \text{where} \quad (a^\dagger)^2 \equiv \frac{8\pi}{3}\rho_{r0}\, a_0^4 = \text{const.} \tag{6}$$

Then show that the solution for the case $k = 0$ is

$$a(t) = (2a^\dagger)^{1/2} t^{1/2}.$$

h. Add the cosmological constant term to the right-hand-side of the Friedmann equation in the matter-dominated era, eq. (1), namely $+\Lambda a^2/3$. What is the solution for $a(t)$ at late times?

6. Light Travel in a Closed Universe

A photon is emitted at a given point (say the north pole of the coordinate system, where the comoving radial coordinate is chosen to be $u_e = 0$) right after the big bang (when $t_e = 0$). It is observed at (t_o, u_o).

a. Recall that the conformal time is defined by $d\eta \equiv dt/a(t)$. Explain why $d\eta = du$ along a photon trajectory (namely $\eta_o - \eta_e = u_o - u_e$).

b. Plot a conformal space time diagram η versus u for the closed universe. Mark along the space axis u the poles and the equator, and along the time axis η the big bang, maximum expansion and big crunch. Plot the world line of the photon.

c. When is the whole universe contained within the horizon of every comoving observer?

d. Can one see events that happened at his position in space at some time in the past?

7. The Horizon and Causality

The comovinig radial distance along a light ray emitted at t_e and observed at t is

$$u = \int_{t_e}^{t} \frac{dt'}{a(t')}.$$

a. Assume a power-law expansion rate $a \propto t^\alpha$ with $0 < \alpha < 1$ (valid when Λ is negligible). Show that a *horizon* exists, namely that when $t_e \to 0$ the distance u traveled by the photon is smaller than a certain maximum value, $u_h(t)$.

b. In the case above, express the horizon distance $r_h(t)$ in proper physical units for the three cases $k = 0, \pm 1$ [use $S_k(u)$].

c. Assume a flat universe ($k = 0$). Compute the comoving horizon distance $u_h(t)$ in the radiation era and in the matter era [what is $a(t)$ in those two cases? assume a negligible Λ].

d. Show that in both cases the horizon radius grows like $r_h \propto t$ (as expected by naive intuition, $r_h \sim ct$).

e. Show that in both cases the mass contained within the horizon grows like $M_h \propto t$.

f. Explain why the above result introduces a *causality* problem concerning the homogeneity of the universe, and in particular given the observed isotropy of the microwave background last-scattering surface.

8. Redshift of Black-Body Radiation

A black-body radiation is characterized by the Planck spectrum with temperature T. The number of photons with a frequency in the interval $(\nu, \nu + d\nu)$, inside a volume V, is:

$$dN = \frac{8\pi\nu^2}{c^3} \frac{V\,d\nu}{e^{\frac{h\nu}{kT}} - 1}.$$

Assume that at time t_1 the universe is filled with a black-body radiation with temperature T_1. Show that at any time t_2 the radiation is still a black body with temperature

$$T_2 = T_1 \frac{a(t_1)}{a(t_2)}.$$

[Consider dN in a comoving box, and recall how V and ν vary as a function of the expansion factor a.]

9. Fluctuation Power Spectrum

a. Explain why $P_k \propto k$ is the scale-invariant power spectrum expected for the initial density fluctuations.

b. Explain why the CDM power spectrum obtains a maximum near a specific wave number $k_{\rm eq}$ and why it asymptotically approaches $P_k \propto k^{-3}$ at $k \gg k_{\rm eq}$. What is the physical origin of the scale corresponding to $k_{\rm eq}$?

c. Express the CDM power spectrum, qualitatively via a schematic plot, in terms of the rms density fluctuation $\delta_{\rm rms}$ on a mass scale M. How do we learn from this function about the hierarchical sequence of formation?

10. Top-Hat Model for Galaxy formation

Assume that a proto-galaxy is a sphere of uniform density ρ_p, whose time evolution can be described by a bound-closed Friedmann model (i.e. a "mini-universe" with $k = 1$ and $\Lambda = 0$). Assume that this sphere is embedded in a background universe which is Einstein-deSitter (i.e. $k = 0$, $\Lambda = 0$), of mean density ρ. We wish to determine the way the density contrast ρ_p/ρ evolves in time. Following is a guide, step by step.

a. From a small density perturbation till maximum expansion

1) Recall that the Friedmann equation of an Einstein-deSitter model in the matter era is

$$\dot{a}^2 = \frac{2a^*}{a} - k, \quad a^* \equiv \frac{4\pi}{3} G\rho_0 a_0^3 ,$$

where ρ_0 and a_0 are the values of the universal density and expansion factor today. Write the implicit solution of the Friedmann equation for the universal expansion factor $a(t)$ in terms of the mass constant a^* and the conformal time η, namely write the expressions for $a(\eta)$ and $t(\eta)$. Do the same for the perturbation, where you denote the corresponding quantities as a_p, a_p^*, η_p, etc.

2) Relate the solutions inside the perturbation and in the background by demanding that the physical time t is the same in both. Use this to relate η to η_p, and then to express a in terms of η_p (rather than η). Recall that we defined $a^* \propto \rho_0 a_0^3 = \rho a^3$ (and a_p^* in analogy), and show that

$$\frac{\rho_p}{\rho} = \frac{9(\eta_p - \sin\eta_p)^2}{2(1 - \cos\eta_p)^3}.$$

3) Define the density perturbation by

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho_p - \rho}{\rho},$$

and use Taylor expansions to show that in the linear regime, when the perturbation is small, $\delta\rho/\rho \ll 1$, namely at early times, $\eta_p \ll 1$, the perturbation growth rate is

$$\frac{\delta\rho}{\rho} \propto t^{2/3}.$$

Compare to what is obtained using linear perturbation analysis in the general (not necessarily spherical) case.

4) Show that at maximum expansion, when the perturbation turns around, the density contrast is

$$\frac{\rho_p}{\rho} = \frac{9\pi^2}{16} \simeq 5.5$$

Note that this is true for any spherical perturbation, no matter when it reaches its maximum expansion.

b. Dark-matter collapse

1) Let the mass inside the perturbation be M, and its radius at maximum expansion be R_{max}. Assume that the kinetic energy at maximum expansion is zero (namely no no-radial motions). Assume that the collapse ends in virial equilibrium, where the kinetic energy equals half the potential energy (absolute value):

$$V^2 = \frac{GM}{R_{vir}}.$$

Use energy conversation during the collapse of dark matter to show that

$$R_{vir} = \frac{1}{2} R_{max}.$$

What is the corresponding growth of density inside the halo between maximum expansion and virialization?

2) What is the density contrast in the virialized halo relative to the background cosmological density at the time of virialization? In addition to the two factors already computed above, we have to include the decrease of the cosmological density between the time of maximum expansion (t_{max}) and the time of virialization (t_{vir}). Take this time to be roughly the time of collapse of the closed "mini-universe", namely when $\eta_p = 2\pi$. Show that the density contrast at virialization is

$$\frac{\rho_p}{\rho} \simeq 176.$$

c. The epoch of galaxy formation

1) Let the observed mean density in a galactic halo be ρ_{vir}, when the cosmological density today is ρ_0. Based on the above computation, what is the epoch of formation (namely virialization) of this halo? Express it in terms of redshift the z_{vir} (recall $1 + z = a_0/a$), and alternatively in terms of time t_{vir}/t_0.

2) Express ρ_0 in terms of Ω_m and the Hubble constant h (where $H_0 \equiv 100h\,\mathrm{km\,s^{-1}Mpc^{-1}}$). Show that

$$(1+z)_{vir} \simeq 6 \left(\frac{\rho_{vir}}{10^{-24}\, g\, cm^{-3}} \right)^{1/3} (\Omega_m h^2)^{-1/3}.$$

3) A halo is observed to have a flat rotation curve with velocity $V = 220\,\mathrm{km\,s^{-1}}$ and a virial radius of $R = 100\,h^{-1}\mathrm{kpc}$. What can we say about its formation epoch?

4) The gas loses energy by radiation and by dissipation during the collapse. By observing the density of the gas (and stellar) component today, what can we say about the epoch of galaxy formation?