COSMOLOGY
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The Metric of a Homogeneous and Isotropic Universe

We chose to express the metric of space in spherical coordinates about a comoving observer, where $u = r/a(t)$ is the comoving radial coordinate. The isotropy implies that

$$d\omega^2 = du^2 + \sigma^2(u)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where $\sigma(u)$ is some function of u only. We wish to determine what functional form is allowed for $\sigma(u)$

In Eucleadian space $\sigma(u) = u$. This should be true also in curved space for small u.

For a general u, consider for simplicity the equatorial plane, $\theta = \pi/2$, and the following diagram:

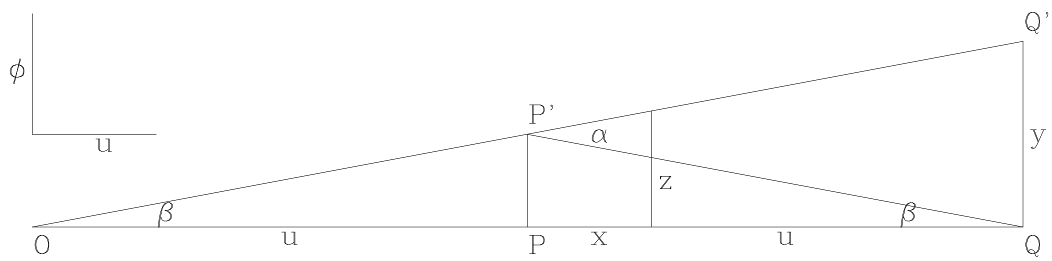

Figure 1: A schematic diagram in the u-ϕ plane

The origin is at O, and the points P and Q at radial distances u and $2u$ respectively. Let the angles α and β be small. Therefore we can write y in two ways using the local flatness:

$$y = \sigma(2u)\beta = \sigma(u)\alpha. \tag{1}$$

Let x be a small interval as shown. Then we can express z in two ways:

$$z = \sigma(u + x)\beta = \sigma(x)\alpha + \sigma(u - x)\beta. \tag{2}$$

We multiply eq. 2 by $\sigma(u)/\beta$ and get using eq. 2

$$\sigma(u + x)\sigma(u) = \sigma(x)\sigma(2u) + \sigma(u - x)\sigma(u).$$

Now we take a derivative with respect to x (mark it with a prime), and obtain at $x = 0$

$$2\sigma'(u)\sigma(u) = \sigma'(0)\sigma(2u).$$

Because the space is assumed to be locally flat, $\sigma(u) \to u$ when $u \to 0$, so $\sigma'(0) = 1$. We finally obtain a differential equation for $\sigma(u)$:

$$2\sigma'(u)\sigma(u) = \sigma(2u).$$

There are 3 solutions to this equation, which we mark by $k = 0, \pm 1$:

$$\sigma(u) = S_k(u) \equiv \begin{cases} \sin(u) & k = 1 \\ u & k = 0 \\ \sinh(u) & k = -1 \end{cases} .$$

We already know that $\sigma(u) = u$ is a solution, and one can easily convince oneself that the other two are "kosher" solutions by substituting them in the differential equation. These are the only solutions. The parameter k stands for the curvature of space.

A way to obtain these solutions without guessing is by writing $\sigma(u)$ as a series expansion

$$\sigma(u) = u + a_3 u^3 + a_5 u^5 + ...$$

The first term must be u because of the local flatness. Once we decide about a_3, the rest of the coefficients are determined by substituting in the differential equation. Depending on the sign of a_3 we obtain the familiar Taylor series for either $A\sin(u/A)$ or $A\sinh(u/A)$. The constant A can be absorbed in the units by which we measure u.

The line element in physical, spherical coordinates, for a homogeneous and isotropic space, is therefore

$$d\ell^2 = a^2(t)\,[du^2 + S_k^2(u)\,d\gamma^2],$$

where $d\gamma$ is the usual area element on a unit sphere, $d\gamma^2 = d\theta^2 + \sin^2\theta\,d\phi^2$, and the function $S_k(u)$ is one of the three possibilities above.