A water bag theory of autoresonant Bernstein-Greene-Kruskal modes

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The adiabatic water bag theory describing formation and passage through phase-space of driven, continuously phase-locked (autoresonant) coherent structures in plasmas [L. Friedland *et al.*, Phys. Rev. Lett. **96**, 225001 (2006)] and of the associated Bernstein-Greene-Kruskal (BGK) modes is developed. The phase-locking is achieved by using a chirped frequency ponderomotive drive, passing through kinetic Čerenkov-type resonances. The theory uses the adiabatic invariants (conserved actions of limiting trajectories) in the problem and, for a flat-top initial distribution of the electrons, reduces the calculation of the self-field of the driven BGK mode to solution of a few algebraic equations. The adiabatic multiwater bag extension of the theory for applications to autoresonant BGK structures with more general initial distributions is suggested. The results of the theories are in very good agreement with numerical simulations. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2771515]

I. INTRODUCTION

Kinetic Bernstein-Greene-Kruskal (BGK) modes in collisionless plasmas were predicted theoretically 50 years ago in Ref. 1. Nevertheless, since Čerenkov-type resonant particles played an important role in the BGK modes, implementation of these nonlinear, dissipationless waves was difficult and required shaping resonant regions of velocity distributions of plasma species. After the BGK discovery, O'Neil[2] predicted that Landau damped plasma waves relax into a BGK equilibrium. Therefore, a natural path to formation of BGK modes used an impulsive excitation-relaxation approach, see, for example, Refs. 3–5. The BGK structures may also emerge due to instabilities, as shown in Refs. 6 and 7. These *passive* routes to formation of the modes were sensitive to initial conditions and could involve a violent stage, leading to poor controllability of the excited BGK structures.

Recently, a different, *active* approach to excitation of the BGK modes was suggested involving driving the plasma by a small amplitude chirped frequency wave passing through kinetic resonances. The idea was first implemented in nonneutral plasmas (trapped, magnetized electron clouds), see Refs. 8–10, using the bounce-type resonance. Recently, a similar approach was suggested in Ref. 11 in quasineutral plasmas via the Čerenkov resonance. The approach involved driving the plasma electrons by a ponderomotive wave and a slow passage through resonances with different particles having flat-top electron velocity distribution initially. By starting with the driving wave phase velocity outside the initial distribution and gradually lowering the phase velocity, stable voids (holes) in phase-space were formed, as illustrated in Fig. 1. The voids were phase-locked to the drive and the electrostatic field associated with the holes constituted a continuously phase-locked (autoresonant) BGK mode in the plasma. The theory of these excitations in Ref. 11 used a fluid-type approach in the initial excitation stage [Fig. 1(a)] and a simplified kinetic theory in dealing with fully developed holes [Fig. 1(c)], but encountered difficulty in describing transition of the hole through the plasma boundary in phase-space [Fig. 1(b)]. In the present work we develop the adiabatic water bag theory of autoresonant phase-space holes and present a uniform description of all stages of excitation in Fig. 1. The use of the water bags idea for studying coherent phase-space structures in collisionless plasmas, see Refs. 6, 12, and 13, started soon after the discovery of the BGK modes. The method relies on the incompressibility of the self-interacting phase-space fluid to describe structures with sharp boundaries in phase-space and, therefore, is most suitable for studying evolution as shown in Fig. 1. Our presentation will be as follows: In Sec. II we will include driving perturbation in the water bag model and describe *driven*, phase-locked water bag equilibria in the case of fixed driving parameters. Later, in Sec. III, using adiabatic invariants in the problem, we will generalize the theory to describe all stages of evolution of driven, slowly varying, autoresonant BGK structures of the type shown in Fig. 1. We will see that the theory allows reduction of the case of a flat-top initial distribution to the solution of a few (three or four, depending on the stage of the evolution) algebraic equations. In Sec. IV, based on a similar reduction, we will generalize to arbitrary initial distributions via a multiwater bag approach. Finally, in Sec. V, we will present our conclusions.

II. STATIONARY, DRIVEN WATER BAG EQUILIBRIA

Consider a one–dimensional (1D), L-periodic, *driven* Vlasov-Poisson system[11]:

$$f_t + uf_x - (E + E_d)f_u = 0, \quad E_x = 1 - \int f du, \tag{1}$$

where $f(x,u,t)$ $[f(x,u,t)=f(x+L,u,t)]$ describes electron velocity distribution in a plasma having a uniform initial electron density n_0, ions of the same density are assumed stationary, and $E(x,t)$ $[E(x,t)=E(x+L,t)]$ is the self-consistent electrostatic wave field. We also assume the presence of the external driving wave $E_d=\varepsilon \sin \Psi_d$ (due to the

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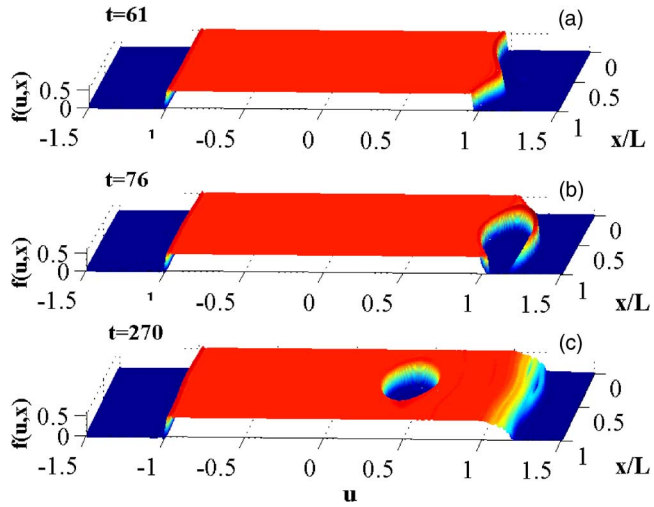

FIG. 1. (Color online) The formation of the autoresonant electron phase-space hole (from Ref. 11): (a) a surface wave in the phase-space fluid, (b) emergence of a void, and (c) a fully-developed autoresonant hole.

ponderomotive effect of beating two transverse laser waves, for example) having amplitude ε, phase $\Psi_d = kx - \int \omega_d(t)dt$, wave vector $k = 2\pi/L$, and slowly *chirped* frequency $\omega_d(t)$. We use dimensionless variables and parameters in Eq. (1), i.e., express u, x, and t in units of the characteristic (thermal) velocity u_{th} of the electrons, the Debye length $\lambda_D = u_{\text{th}}/\omega_p$, and the inverse plasma frequency ω_p, respectively. The distribution function is rescaled by n_0/u_{th} and the dimensionless fields are in units of $mu_{\text{th}}\omega_p/e$. Numerical solutions of this system with a flat-top initial electron distribution of height 1/2 between $u = \pm 1$ and using driving frequency $\omega_d(t) = \omega_0 - \alpha t$ having constant chirp rate α were studied in Ref. 11 with a typical evolution illustrated in Fig. 1. The idea of transition from the Vlasov to the water bag description in this system is based on the incompressibility of the electron phase-space fluid, so that the electron distribution function remains constant in some region of phase-space during the evolution, but the sharp boundaries of the electron phase-space distribution evolve in time to form driven, phase-locked equilibria with or without a hole. This means that one can fully describe the evolution of the plasma, by characterizing the dynamics of the electron phase-space boundaries. We proceed to study these dynamics next.

We will make two important assumptions in our theory. Both are inspired by the results of our simulations. The first assumption is the phase-locking in our driven system, i.e., that the self-electric field waveform has the same phase Ψ_d as the driving wave continuously despite the variation of the driving frequency. Secondly, we will assume that the self-field is eikonal, i.e., $E = a(t)\sin\Psi_d$, with a slowly varying amplitude $a(t)$. Our simulations in Ref. 11 showed that higher harmonics of the self-field were small in the studied parameter range. We will discuss the validity of this assumption later in the theory. Finally, in contrast to Ref. 11, we will not assume a constant driving frequency chirp rate, but require slowness of variation of $\omega_d(t)$ only and will not limit the discussion to constant driving amplitude ε, allowing it also to be a slow function of time. In other words, we will assume that the relative change of ε, ω_d, and a (the amplitude of the self-field) during one characteristic period T of the underlying dynamics (see below) is small. The goal is to describe the adiabatic evolution of our system such that at any given time t, the plasma state is close to one of the allowed equilibria (in an imaginary plasma) with ε, ω_d having values at all times as those in our real plasma at time t. Thus, we consider possible *stationary,* driven, constant amplitude, phase-locked water bag equilibria at fixed values of the driving parameters, ε and ω_d first.

Consider the dynamics of individual plasma electrons in combined, phase-locked driving and self-fields, $E_{\text{tot}} = a' \sin(kx - \omega_d t)$, where $a' = a + \varepsilon$ and ε, ω_d, and a are constant. The motion of the electrons in this case is described by

$$\ddot{x} = -a' \sin(kx - \omega_d t). \tag{2}$$

By transformation to the frame moving with the phase velocity $u_p = \omega_d/k$, i.e., introducing $x' = x - u_p t$, we obtain the pendulum equation

$$\ddot{x}' = -a' \sin(kx'). \tag{3}$$

The latter yields the conserved energy function ($u' = \dot{x}'$),

$$H(u',x') = \frac{1}{2}u'^2 - \frac{a'}{k}\cos(kx'). \tag{4}$$

Then, for a given H, one can express the electron velocity $u = u_p + u'$ in the rest frame as a function of x', i.e.,

$$u = u_p \pm \left[2\left(H + \frac{a'}{k}\cos(kx')\right)\right]^{1/2}. \tag{5}$$

Next, we choose two values $H_{1,2} > a'/k$, $H_1 < H_2$, i.e., two untrapped trajectories in the phase-space, assume that the velocities u' (in the moving frame) on these trajectories are *negative* and that the whole electron phase-space fluid, at all times, is confined between the two trajectories

$$u_{1,2}(x') = u_p - \left[2\left(H_{1,2} + \frac{a'}{k}\cos(kx')\right)\right]^{1/2}. \tag{6}$$

This yields a driven water bag equilibrium illustrated in Fig. 2(a) (for parameters $k=3$, $\varepsilon=0.01$, $u_p=1.14$, $H_1=0.0075$, and $H_2=2.3$), resembling the plasma state in simulations in Fig. 1(a).

Note that we needed the self-field amplitude a in the illustration in Fig 2(a). For finding a, we used the Poisson equation

$$E_{x'} = 1 - \tfrac{1}{2}(u_1 - u_2), \tag{7}$$

which, after taking the Fourier transform, yielded

$$ka = F_2 - F_1, \tag{8}$$

where, $F_i = 1/L\int_{-L/2}^{L/2} u_i \cos(kx')dx'$ or, using Eq. (6),

$$F_i = \frac{2}{3\pi}\sqrt{\frac{2a'(g-1)}{k}}[(g+1)K(\mu) - gE(\mu)]. \tag{9}$$

Here $g = (H_i k)/a' > 1$, $\mu = 2/g - 1$, and K, and E are the complete elliptic integrals of the first and second kind, respectively. Given $H_{1,2}$, ε, and u_p, (8) is an algebraic equation for the self-field amplitude a. We found $a = 0.0089$ for param-

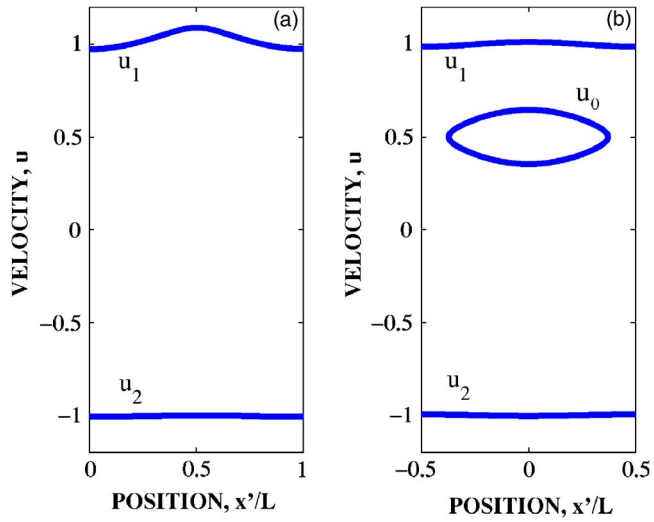

FIG. 2. (Color online) The stationary (in the moving frame), driven water bag equilibria: (a) a surface wave in the phase-space fluid, and (b) a phase-locked hole in phase-space. Velocities u_i are the limiting trajectories in the water bag model.

eters in Fig. 2(a) and used this value for plotting the two limiting trajectories in the figure.

Note that for $\varepsilon=0$, by replacing ω_d by Ω, one can view Eq. (8) as defining the nonlinear dispersion relation $\Omega=\Omega(k,a)$ of a *free*, stationary waveform associated with the water bag defined by the values of $H_{1,2}$. Also, note that the calculation above violates the neutrality and, therefore, needs a correction. Indeed, the averaged electron density in the model is

$$\langle n_e\rangle = \frac{1}{2L}\int_{-L/2}^{L/2}(u_1-u_2)dx' = J_1 - J_2, \tag{10}$$

where $J_i=1/2L\int_{-L/2}^{L/2}u_i dx'$, yielding, upon integration,

$$J_i = \frac{u_p}{2} - \frac{1}{\pi}\sqrt{\frac{2a'(g+1)}{k}}E\left(\frac{2}{1+g}\right). \tag{11}$$

This $\langle n_e\rangle$ will be different from unity (the ion density) in general. Nevertheless, the problem can be fixed, by choosing the background ion density to be $n_i=\langle n_e\rangle$.

Finally, we can test our sine-wave assumption for E by estimating the second harmonics amplitude of the electric field from the Poisson equation,

$$2ka_2 = G_2 - G_1, \tag{12}$$

where $G_i=1/2L\int_{-L/2}^{L/2}u_i\cos(2kx')dx'$. Here, we can still use expressions (6) for u_i, where one adds the second harmonic contribution $a_2/2k\cos(2kx')$ in potential energy due to the self-field. Then, Eq. (12) becomes an algebraic equation for a_2, which can be solved by iterations. In the example in Fig. 2(a) we have found $a_2=-0.00033$, as compared to $a=0.0089$. If one includes this second harmonic contribution in the potential energy due to the wave and recalculates a, one finds only 2% correction. This justifies our sine-form assumption for the electric field in calculating the amplitude a of the first harmonic.

Similarly to the case in Fig. 2(a), one can also construct driven, stationary, phase-locked water bag solutions with a hole inside the electron phase-space distribution. Indeed, let us choose three values of the energy, $H_{1,2}>a'/k$ (two untrapped trajectories) and $H_0<a'/k$ (a trapped trajectory), assume that the velocities u' of the untrapped trajectories in the moving frame are positive or negative for H_1 or H_2, respectively, and that the electron phase-space fluid at all times is confined between the open trajectories

$$u_1(x') = u_p + \left[2\left(H_1 + \frac{a'}{k}\cos(kx')\right)\right]^{1/2}, \tag{13}$$

$$u_2(x') = u_p - \left[2\left(H_2 + \frac{a'}{k}\cos(kx')\right)\right]^{1/2}, \tag{14}$$

and the closed trajectory

$$u_0(x') = u_p \pm \left[2\left(H_0 + \frac{a'}{k}\cos(kx')\right)\right]^{1/2}. \tag{15}$$

This yields the driven water bag state shown in Fig. 2(b), which is similar to that seen in simulations in Fig. 1(c). For this illustration, we again needed the self-field amplitude a. The latter was obtained by taking the Fourier transform of the Poisson equation, yielding

$$ka = F_2 + F_1 + F_0, \tag{16}$$

where, as before, $F_{1,2}$ are given by Eq. (9), while $F_0=\int u_0\cos(kx')dx'$ with the integration around the closed limiting trajectory (15) and can also be expressed in terms of the elliptic integrals

$$F_0 = \frac{4}{3\pi}\sqrt{\frac{a'}{k}}[2g_0E(\mu_0) + (1-g_0)K(\mu_0)], \tag{17}$$

where $g_0=(H_0k)/a'<1$ and $\mu_0=1+g_0/2$. We have solved Eq. (16) by iterations with the parameters of Fig. 2(b) and found $a=0.00328$. Again, for the quasineutrality, one must adjust the ion density to $n_i=\langle n_e\rangle$, i.e., assume

$$n_i = \frac{1}{2L}\left[\int_{-L/2}^{L/2}(u_1-u_2)dx' - \int u_0 dx'\right] = J_1 - J_2 - J_0, \tag{18}$$

where the $J_{1,2}$ are given by Eq. (11) with a plus sign in the second term in the expression for J_1, while $J_0=1/2L\int u_0dx'$ or, explicitly,

$$J_0 = \frac{4\sqrt{a'}}{\pi}[E(\mu_0) - (1-\mu_0)K(\mu_0)]. \tag{19}$$

Finally, we can estimate the second harmonics amplitude a_2 from the Poisson equation [compare to Eq. (12)]

$$2ka_2 = G_2 - G_1 + G_0 \tag{20}$$

where, as before, $G_{1,2}=1/2L\int_{-L/2}^{L/2}u_i\cos(2kx')dx'$, while $G_0=1/2L\int u_0\cos(2kx')dx'$ with the integration around the closed limiting trajectory. In the example in Fig. 2(b) we found $a_2=0.0014$ as compared to $a=0.0328$, again justifying our sine-wave approximation in calculating a.

III. THE ADIABATIC APPROXIMATION

In the previous subsection we have seen construction of the stationary, driven, phase-locked water bags, completely prescribed by the values of the energies $H_{0,1,2}$ on the limiting trajectories and *fixed* driving wave parameters ω_d, k, and ε. At this stage, the theory is still inapplicable to the adiabatically driven case as seen in simulations in Fig. 1, where ω_d and, as a result, the self-field amplitude a are slow functions of time. In this case, the energies $H_{0,1,2}$ also become functions of time. Furthermore, as the driving phase velocity $u_p(t)=\omega_d(t)/k$ decreases in Fig. 1, one cannot adjust the ion density during the evolution in order to preserve the neutrality, the plasma must automatically remain neutral at all times. All these difficulties will be resolved in this section, by using the adiabatic invariants in the problem.

We will consider the adiabatic problem by starting from the mixed variable Hamiltonian representation of the electron dynamics in combined chirped frequency driving and self-consistent electrostatic waves, i.e., use the velocity u in the rest frame and the coordinate of the particle, $x'=x-\int u_p(t)dt$, in the moving (decelerating) frame, as our canonical variables. Equations of motion in this case are

$$\dot{u}=-[a(t)+\varepsilon]\sin(kx'), \quad \dot{x}'=u-u_p(t). \tag{21}$$

The associated Hamiltonian is

$$H(u,x',\lambda(t))=\frac{1}{2}[u-u_p(t)]^2-\frac{[a(t)+\varepsilon]}{k}\cos(kx'), \tag{22}$$

where $\lambda(t)$ represents the set of slowly varying parameters in the problem, i.e., a, u_p, and, possibly, ε. Generally, all oscillatory Hamiltonian problems governed by Hamiltonians of type $H(p,q,\lambda(t))$ where $\lambda(t)$ are slowly varying parameters possess an adiabatic invariant, see, for example, Ref. 14, i.e., if one formally inverts $H=H(p,q,\lambda)$ writing it as $p=p(q,H,\lambda)$, then the action $I(H,\lambda)=(2\pi)^{-1}\int p(q,H,\lambda)dq$, where the integration is performed over one oscillation period with λ fixed at a given time, is adiabatically conserved. In other words, as λ is slowly varying in time, the energy H of the trajectory also varies, but, nevertheless, the action I is conserved, provided parameters λ do not change over one oscillation period significantly. Importantly, the action is conserved on both trapped and untrapped trajectories.

In our driven plasma case, $p=u$ and one finds by inspection that the inverted expression $p=p(q,H,\lambda)$, coincides with expressions (6) or (13)–(15) for the velocities of our limiting trajectories. Therefore, the integrals $J_i(H,\lambda)$ defined in Eqs. (11) and (19) are adiabatic invariants. This observation is sufficient for solving our adiabatically driven problem. Indeed, if one starts with $u_p \gg 1$, i.e., far outside the electron velocity distribution, and slowly increases the driving amplitude by starting from zero (i.e., initially, $\varepsilon=a=0$), the initial limiting trajectories will be the unperturbed boundaries of the phase-space fluid, $u_{1,2}=\pm 1$. Then, initially, $J_{1,2}=\pm\frac{1}{2}$ [recall the definition (11)]. Later, as the parameters vary, say ε reaches a constant value and one slowly chirps down the driving frequency, $J_{1,2}$ (the actions on the untrapped limiting trajectories) remain constant. Therefore, Eq. (11) yields two algebraic equations for the three unknowns, $H_{1,2}$ and a, as functions of the slowly varying phase velocity $u_p(t)$. Adding the Poisson equation (8), we arrive at a closed system of three algebraic equations for the three variables, and the solution of this system yields the desired slow evolution of the self-consistent field amplitude $a=a(u_p(t))$. Note that during the adiabatic evolution, the averaged electron density remains at its initial value of unity (the plasma remains neutral), because the integrals $J_{1,2}$ in Eq. (10) remain constant. Note also that for quasineutrality during the excitation process one only needs the constancy of J_1-J_2, representing the conservation of the electron phase-space volume. The conservation of the two actions $J_{1,2}$ separately comprises a stronger statement. The calculation above describes the initial excitation stage, when both limiting trajectories correspond to the untrapped motion in the combined driving and self-fields [see Fig. 1(a)]. The transition through the phase-space fluid boundary and the subsequent emergence and evolution of a phase-space hole will be discussed next.

The transition through the boundary occurs as u_p decreases and the energy H_1 of the limiting trajectory u_1 [see Eq. (6) and the illustration in Fig. 2(a)] approaches a'/k, i.e., u_1 approaches the separatrix:

$$u_1(x') \to u_s = u_p + \left[2\frac{a'}{k}(1+\cos(kx'))\right]^{1/2}. \tag{23}$$

Our adiabatic analysis fails on the separatrix because of the infinitely large period of the trajectory. However, if the variation of u_p is sufficiently slow, one can approach a close vicinity of the separatrix without breaking the adiabatic invariance. One can estimate the value of the driving wave phase velocity u_p^s at this critical stage by calculating $J_1^s \approx \frac{1}{2}(u_p^s - 4/\pi\sqrt{a_s'/k})$ at the separatrix and using the adiabatic invariance, $J_1=\frac{1}{2}$. This yields the critical phase velocity $u_p^s \approx 1 + 4/\pi\sqrt{a_s'/k}$. As the phase velocity of the drive continues to decrease, some of the phase-space fluid flows around the separatrix [see simulations in Fig. 1(b)] and reconnects with the rest of the fluid due to a small dissipation, introduced in the simulations for dealing with the singularities of the Vlasov code at sharp phase-space boundaries. After the reconnection, the electron phase-space hole is formed, having nearly the form of the separatrix and energy $H_0 \approx H_1 \approx a_s'/k$. The integral J_0 associated with this hole is $J_0^s = 4/\pi\sqrt{a_s'/k}$ and remains invariant, if the hole moves away from the separatrix into the bulk of the distribution. Note that the initial values of H_0 and J_0^s, just after formation of the hole, are fully determined by the value of the self-field amplitude a^s at the transition, while J_1 experiences a jump with the formation of the hole and becomes $J_1^s \approx \frac{1}{2}+J_0^s$ at the transition point, but J_2 remains unchanged. These actions comprise a set of initial data for following the hole into the bulk of the distribution by solving the set of algebraic equations, $J_1=J_1^s$, $J_2=-1/2$, $J_0=J_0^s$, and the Poisson Eq. (16) for $H_{0,1,2}$ and a as functions of $u_p(t)$, thus completing the solution of the problem. Importantly, as before, the conservation of the adiabatic invariants guarantees the continuing quasineutrality of the plasma with the hole inside, since $\langle n_e \rangle = J_1 - J_2 - J_0 = 1$ continuously. Furthermore, as the hole is formed at the plasma phase-fluid boundary, the local plasma density remains nearly un-

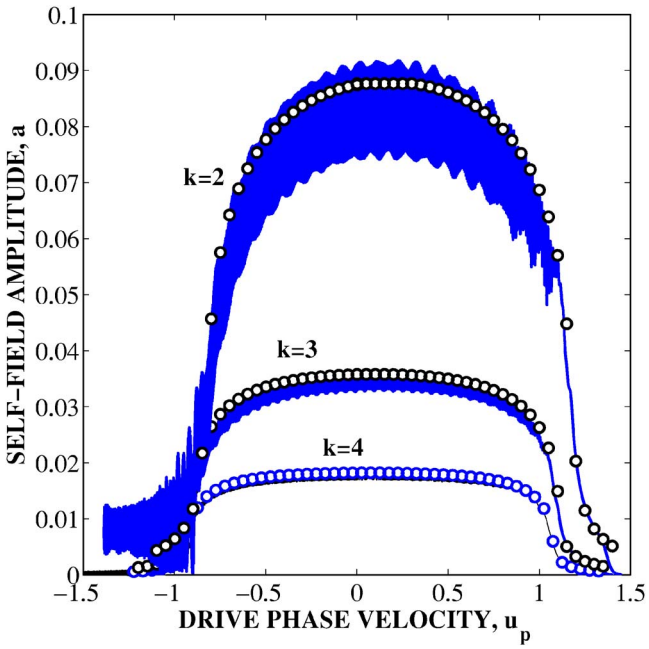

FIG. 3. (Color online) The amplitude a of the driven BGK mode versus the driving wave phase velocity $u_p(t)$ for different $k=2$, 3, and 4, as given by the adiabatic water bag theory (circles) and simulations (solid lines).

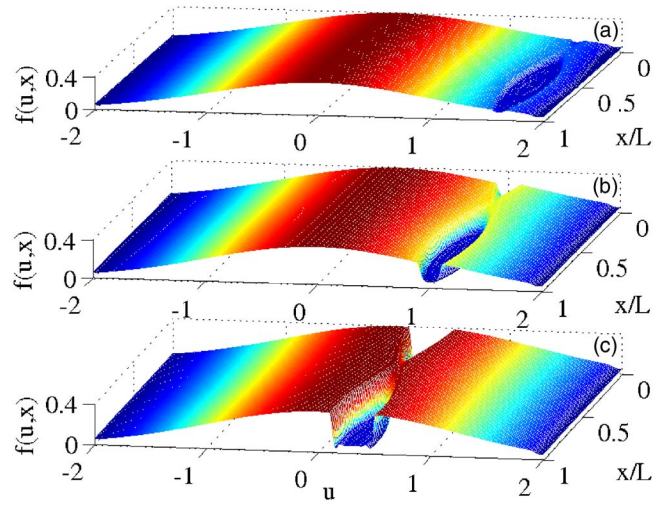

FIG. 4. (Color online) The driven phase-space depression formed by a chirped frequency driving wave in the initially Maxwellian distribution. The structures are shown at three different evolution times, as the driving wave phase velocity assumes the values (a) $u_p=1.6$, (b) $u_p=1.0$, and (c) $u_p=0.3$.

changed, i.e., $(u_1-u_2)_{t_s-0}\approx(u_1-u_2-\Delta u_0)_{t_s+0}$. Then, via the Poisson equation, both the electric field and its spatial derivative are continuous, as the hole emerges in the electron phase-space.

We have performed calculations for verifying predictions of the adiabatic water bag theory presented above. Figure 3 shows the comparison between the results for $a=a[u_p(t)]$ from the theory and simulations. We used $\varepsilon=0.015$, 0.01, and 0.0075 (after the initial ramp up from zero) for three values of $k=2$, 3, and 4, respectively. The chirp rate was $\alpha=0.003$, so that $\alpha/(\epsilon k)=0.1$ in all cases in Fig. 3. One can see a very good agreement between the theory and simulations in the figure. Note, that because of time reversibility, the theoretical results for $a(u_p)$ are symmetric around u_p value of J_0^s. This also means that after the passage of the hole through the whole phase-space distribution, the latter is shifted by J_0^s in the velocity space yielding a net current. Thus, chirped frequency ponderomotive wave passage through the electron velocity space may serve as an efficient current drive.

IV. GENERALIZATION TO ARBITRARY DISTRIBUTIONS

Coherent phase-space structures can be formed by a chirped frequency ponderomotive drive when starting from other initial phase-space distributions. For example, Fig. 4 shows the evolution of the electron distribution function in simulations similar to those in Fig. 1 (we used the same Vlasov code as in Ref. 11), but starting with the Maxwellian, $f=(2\pi)^{-1/2}\exp(-u^2/2)$ and using $k=4$, $\alpha=0.003$, and $\epsilon=0.0075$. One observes formation of a growing depression in the electron phase-space distribution with the local minimum moving in the velocity space towards the center of the distribution, as the driving frequency decreases and assumes the values 1.6, 1.0, and 0.3 at three different time moments in Figs. 4(a)–4(c), respectively. These more complex structures can be studied by utilizing a multiwater bag model, as described in the following. We discretize the velocity space and view the initial distribution as a superposition of many flat-top layers of thickness $|df/du_i|\Delta u$ each, where $\Delta u=u_{i+1}-u_i$. The ith layer is now driven by the combination of the ponderomotive and self-field and evolves similarly to the single layer described above. In other words, every layer can be viewed as a flat-top distribution confined between two or three adiabatically varying limiting trajectories (depending on whether the hole is outside or inside the layer) characterized by the slow parameters in the problem, u_p, $a(t)$, and the energies $H_{1,2}^i$ or $H_{0,1,2}^i$. In addition, each limiting trajectory of the ith layer has its adiabatic invariants, $J_{1,2}^i$ or $J_{0,1,2}^i$. The latter are determined by the initial conditions and the jump condition if the hole enters the layer as described above. Then, by solving a set of algebraic equations $J_{1,2}^i(H_{1,2}^i,u_p,a)=\text{const}$ or $J_{0,1,2}^i(H_{0,1,2}^i,u_p,a)=\text{const}$ for the water bags without or with the hole, respectively, in combination with the Poisson equation

$$ka = S_{\text{out}} + S_{\text{in}}, \tag{24}$$

where [compare to Eqs. (8) and (16)]

$$S_{\text{out}} = 2\sum_{i'} |df/du_{i'}|\Delta u(F_2^{i'} - F_1^{i'}) \tag{25}$$

and

$$S_{\text{in}} = 2\sum_{i''} |df/du_{i''}|\Delta u(F_2^{i''} + F_1^{i''} + F_0^{i''}) \tag{26}$$

yields the desired evolution of $H_{1,2}^{i'}$, $H_{0,1,2}^{i''}$, and a versus $u_p(t)$. The indices i' and i'' in Eqs. (25) and (26) correspond to all layers without or with the holes, respectively, inside the corresponding water bag, while $F_{0,1,2}^i$, as for the theory of a single water bag above, are the Fourier transforms of the

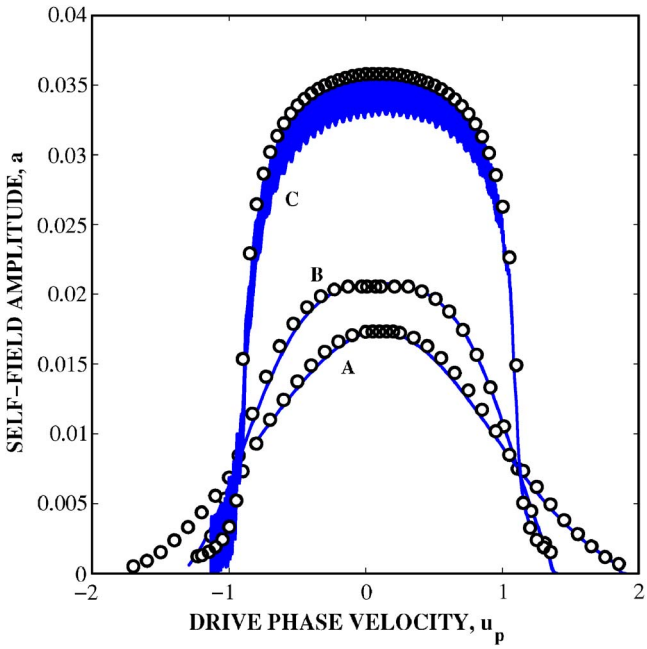

FIG. 5. (Color online) The amplitude a of the driven BGK mode versus the driving wave phase velocity $u_p(t)$ for (A) the Maxwellian, $f \propto \exp(-u^2/2)$, (B) the super-Maxwellian, $f \propto \exp(-u^4/2)$, and (C) the flat-top initial distributions. Simulation results are shown by solid lines, while circles represent the water bag theory.

corresponding limiting velocities, $F_i = \int_{-L/2}^{L/2} u_i \cos(kx')dx'$. We have performed calculations based on this multiwater bag model for two different initial distributions, i.e., a Maxwellian $f \propto \exp(-u^2/2)$, and a super-Maxwellian, $f \propto \exp(-u^4/2)$. The results are compared with simulations in Fig. 5. The figure shows the evolution of the amplitude a of the excited driven BGK structures for parameters $k=3$, $\alpha=0.003$, and $\varepsilon=0.01$. Again, one observes an excellent agreement between the theory and simulations. The results for the flat-top distribution (see Fig. 3) are also shown in Fig. 5 for comparison. This completes our analysis of the chirped, driven coherent BGK structures within the adiabatic water bag theory.

V. CONCLUSIONS

(a) Adiabatically evolving, coherent phase-locked structures in phase-space and the associated BGK modes can be formed in plasmas by driving plasma electrons by a small amplitude, chirped frequency ponderomotive force. In the developed evolution stage these structures comprise electron phase-space distribution depressions drifting in the velocity space with the rate of variation of the phase velocity of the driving perturbation. Numerical simulations showed robust stability of these phase-locked BGK structures.

(b) We have developed the adiabatic water bag theory of the driven BGK modes emerging due to chirped frequency drives for a flat-top initial electron velocity distribution. The theory used adiabatic invariants in the problem (actions of the limiting trajectories) and allowed calculation of the evolution of the electric field of the BGK mode by solving just a few (three or four) algebraic equations. It yielded a uniform description of the whole BGK mode excitation process including the initial excitation at the electron phase-space boundary, the passage through the boundary, and the emergence of a fully developed electron phase-space hole. The results of the adiabatic theory were in very good agreement with simulations.

(c) We have generalized the theory to different initial velocity distributions by adopting the multiwater bag approach. This generalization viewed the initial electron velocity distribution as a superposition of multiple thin flat-top distribution layers and used the adiabatic invariants associated with each of the layers. The results of the theory were again in very good agreement with simulations and allowed uniform description of the formation of more complex phase-space structures emerging due to the chirp frequency driving.

(d) It seems important to extend the adiabatic water bag theory in the future to studying the numerically observed stability of the driven chirped BGK modes. The inclusion of the self-consistent evolution of the driving field in the theory is also an interesting goal for future research.

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