Equal energy phase space trajectories in resonant wave interactions

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Adiabatic evolution of two and three resonantly interacting wave systems with nonlinear frequency/wave vector shifts is discussed. The corresponding Hamiltonian, depending on the coupling, detuning, and nonlinear frequency shift parameters may have a variable number of fixed points, i.e., the system can experience a topological change of phase space when these parameters vary in time or space. It is shown that the oscillation periods of two equal energy trajectories in these wave systems are equal and the difference between the action integrals of such trajectories is obtained analytically as a function of the system parameters. Based on these findings, a scheme of simultaneous adiabatic variation in the parameters is designed, such that any pair of initially equal energy trajectories continues to have the same energy at later times. These results are generalizations of a previous work [O. Polomarov and G. Shvets, Phys. Plasmas **13**, 054502 (2006)] for a single, resonantly driven wave. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3139263]

I. INTRODUCTION

Resonant excitation of nonlinear waves is important in many plasma physics applications (see, for example, Ref. 1 and references therein). Weakly nonlinear driven wave dynamics in many of these applications is described by the following Hamiltonian:

$$H(I,\theta,t) = b(t)I - c(t)I^2 - a(t)\sqrt{2I}\sin\theta, \tag{1}$$

where I and θ are the canonical action-angle variables, and a, b, and c are parameters, which may vary in time. Physically, $\sqrt{I}$ is proportional to the amplitude of the driven wave, θ is the relative phase between the wave and the drive, a describes the slowly varying driving envelope, b is the driving frequency detuning from the linear resonance, and $2cI$ represents a weakly nonlinear driven wave frequency shift. The same Hamiltonian (1) also describes many other resonantly driven problems in physics, examples ranging from atomic[2] and planetary[3,4] dynamics to fluids,[5] Bose–Einstein condensates[6] and Josephson junctions.[7]

If the variation of parameters in the Hamiltonian (1) is slow, the underlying dynamics can be understood by studying the associated phase space structure for fixed parameters.[8–10] For example, depending on the values of parameters a, b, and c, one may encounter either one (elliptic) fixed point or three (two elliptic, one hyperbolic) fixed points in phase space. Therefore, the topology of the phase space evolution may change considerably by variation of system parameters and one may expect transition from small to large amplitude excitations because of this topological change. We will refer to this type of excitation of nonlinear waves as the *bistability* approach.[1] A different, *autoresonant* approach[11] to excitation of nonlinear waves described by Eq. (1) uses a salient property of the system, under certain conditions, to stay in nonlinear resonance continuously as the driving frequency and, therefore, parameter b in Eq. (1), pass the linear resonance ($b=0$) in the system. Here we will focus on the bistability approach to excitation of nonlinear waves only. A recent discovery in this context was the understanding[1] that two trajectories governed by Eq. (1) corresponding to the same value of Hamiltonian (*sister* trajectories[1]) play a special role when the phase space dynamics changes its topology, as described above. In particular, if the topological change occurs due to an adiabatic variation of the driving parameter a, with b and c remaining constant, two initially equal energy trajectories remain sisters, despite the change in the topology during the evolution. This property of Eq. (1) was used in Ref. 1 to predict the outcome for the excited wave in the bistability approach analytically, when parameter a is slowly switched on and off passing the critical value a_{cr}, where the topological change takes place.

The generic Hamiltonian (1) describes resonant excitation of weakly nonlinear waves by a *prescribed external* driver. However, in many cases, the driver itself is another wave, which may be depleted during the excitation process. Such resonant two-wave interactions are usually called a mode conversion (MC).[12] Similarly, when the driver comprises a beat wave due to two other waves, the process is called a resonant three-wave interaction (R3WI). The two driving (pump) waves in this case may be depleted significantly as the parameters vary in time and/or space. The resonant two- and three-wave interactions play a fundamental role in physics by representing most important linear and lowest order (in terms of wave amplitudes) nonlinear effects in systems approximately described by a linear superposition of discrete waves. For example, an incoming laser beam in a plasma can decay via R3WI into another electromagnetic wave and an ion-acoustic wave (stimulated Brillouin scattering) or an electrostatic plasma wave (stimulated Raman scattering or SRS).[13] In particular, SRS is of great interest as a reflection mechanism in inertial confinement fusion[14–18] or as a mechanism for optical pulse compression in plasma-based Raman amplifiers.[19–22] With all this diversity, both the MC and R3WIs involving weakly nonlinear frequency (wave

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vector) shifts in time [or one-dimensional (1D) space] varying plasmas are described by Hamiltonians similar to Eq. (1), but involve a generalization in the interaction part, i.e.,[12,23]

$$H(I,\theta,t) = b(t)I - c(t)I^2 - a(t)\sqrt{2R(I)} \sin\theta, \tag{2}$$

where $R=I(Q_1-I)$ and $R=I(Q_1-I)(Q_2-I)$ for two- or three-resonantly coupled waves, respectively. Here I is again proportional to the square of the amplitude of one of the waves, while the constants $Q_{1,2}$ represent Manley–Rowe relations of I to the amplitudes of other waves proportional to $(Q_i-I)^{1/2}$. As in Eq. (1), b and c characterize the linear and nonlinear frequency (wave vector) shifts, a is the wave coupling. All these parameters may depend on time (space) in a time dependent (1D nonuniform) plasma case. The Hamiltonian (2) reduces to Eq. (1) if $Q_i \gg I$, i.e., when the pump depletions are negligible. Nevertheless, one must use a more general Hamiltonian in two or three coupled waves systems if a significant pump depletion is expected, e.g., the SRS in a nonuniform plasma.[24,25]

The present work studies a controlled wave excitation in resonant two- and three-wave interactions, via the bistability approach in systems described by Eq. (2). We will develop a new approach in analyzing pairs of sister trajectories in these systems, playing an important role in wave interactions, similar to the case of single driven waves.[1] We will find a scheme of adiabatic variation in parameters a, b, and c in the problem, such that any pair of trajectories having the same energy at a certain time will have the same energies continuously at later times, i.e., remain sister trajectories. This will allow us to predict, for example, the final state of the excited system when the coupling parameter is adiabatically switched on and off and the evolution involves the aforementioned change of topology of the associated phase space. Our presentation will be as follows. Section II will proceed from a numerical example of preservation of sister trajectories under an adiabatic variation of parameters in a three wave system. The phase space structure of these trajectories in one-, two-, and three-wave interactions in the case of constant parameters will be also discussed in Sec. II. We will analyze the evolution of sister trajectories under adiabatic variation of parameters in Sec. III and find conditions for continuous preservation of such pairs of trajectories in different coupled wave systems. Finally, our conclusions will be presented in Sec. IV.

II. SISTER TRAJECTORIES

A. Numerical examples

The dynamics governed by the Hamiltonian H is described by the corresponding evolution equations

$$\frac{dI}{dt} = -\frac{\partial H}{\partial \theta} = a\sqrt{2R(I)} \cos\theta, \tag{3}$$

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial I} = b - 2cI - a\frac{d}{dI}(\sqrt{2R(I)})\sin\theta. \tag{4}$$

For fixed parameters a, b, and c, the phase space trajectories in the (I,θ) plane are the constant energy contours $H(I,\theta)$ $=h$. We define sister trajectories[1] as two trajectories characterized by the same energy h. Our main goal is to show that it is possible to make a simultaneous adiabatic change of parameters a, b, and c so that any pair of trajectories having the same energy at some time will remain sister trajectories at later times. For example, we will see below that in the case of two coupled waves, the trajectories remain sisters continuously for any adiabatic variation of the parameters preserving the ratio $b(t)/c(t)$. This is the generalization of a similar property for a single driven wave.[1] In contrast, for three coupled waves, the condition for having sister trajectories continuously is $a^2(t)/c^2(t)+2b(t)/c(t)=\text{const}$. We illustrate this remarkable property in the case of three coupled waves in Fig. 1, where the evolution of I and h in this case is shown for $a(t)=a_0\exp(-t^2/t_a^2)$, $b(t)=b_0[1-\exp(-t^2/t_b^2)]=[a_0^2/(2c)]\times[1-\exp(-2t^2/t_a^2)]$, $c=\text{const}$, and two initial conditions ($I_{\text{in}}=0.08$, $\theta_{\text{in}}=0$, and $\theta_{\text{in}}=3\pi/2$) yielding a pair of sister trajectories. We use parameters $c=3/16$, $a_0=0.2$, $b_0=a_0^2/(2c)=0.1067$, $t_a/\sqrt{2}=t_b=4000$, $Q_1=1$, and $Q_2=1.4$. One can see in Fig. 1(a) that the difference in the initial conditions results in two different solutions of our time-dependent parameters system. Nevertheless, their time varying energies $h(t)$ [see Fig. 1(b)] remain the same, i.e., the trajectories remain sisters continuously under chosen adiabatic variation of the parameters.

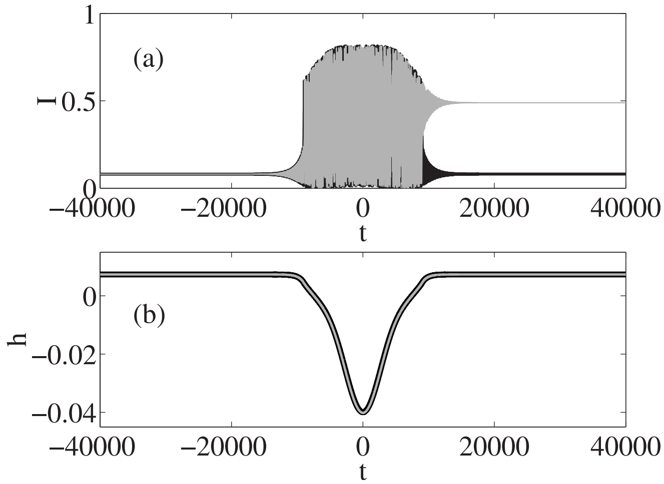

FIG. 1. The evolution of a resonantly coupled three-wave system with time dependent parameters preserving sister trajectories for initial conditions: $I_{\text{in}}=0.08$ and $\theta_{\text{in}}=0$ (dark color) and $\theta_{\text{in}}=3\pi/2$ (light color). (a) The action I vs time. (b) The energy h vs time. The parameters are $a(t)=a_0\exp(-t^2/t_a^2)$, $b(t)=b_0[1-\exp(-t^2/t_b^2)]$, $c=3/16$, $a_0=0.2$, $b_0=a_0^2/(2c)=0.1067$, $t_a/\sqrt{2}=t_b=4000$, $Q_1=1$, and $Q_2=1.4$. In this case, $a^2+2bc=\text{const}$ and the energies of the two different trajectories remain the same at all times.

In Fig. 2 we present a different case in which we again solve the equations for three coupled waves when all parameters, but $b(t)$ are constant: $a=a_0=0.02$, $b(t)=b_0[1-\exp(-t^2/t_b^2)]$, $b_0=50a_0^2/c=0.213$, $c=3/16$, $Q_1=1$, $Q_2=1.4$, $t_b=4000$, $I_{\text{in}}=0.001$, $\theta_{\text{in}}=0$, and $\theta_{\text{in}}=\pi$. The two initially sister trajectories in Fig. 2(a) cease to remain sisters as the energy difference between them develops, as shown in Fig. 2(b).

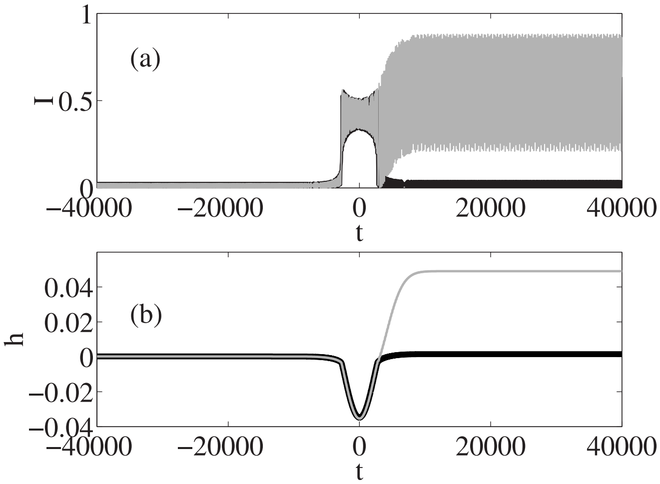

FIG. 2. The evolution of a resonantly coupled three-wave system with different time dependent parameters for initial conditions: $I_{in}=0.001$ and $\theta_{in}=0$ (dark color) and $\theta_{in}=\pi$ (light color). (a) The action I vs time. (b) The energy h vs time. The parameters are $a=a_0=0.02$, $b(t)=b_0[1-\exp(-t^2/t_b^2)]$, $b_0=50a_0^2/c=0.213$, $c=3/16$, $Q_1=1$, $Q_2=1.4$, and $t_b=4000$. In this case, a^2+2bc is not kept constant and the trajectories stop being sisters after parameter $b(t)$ returns to its initial value.

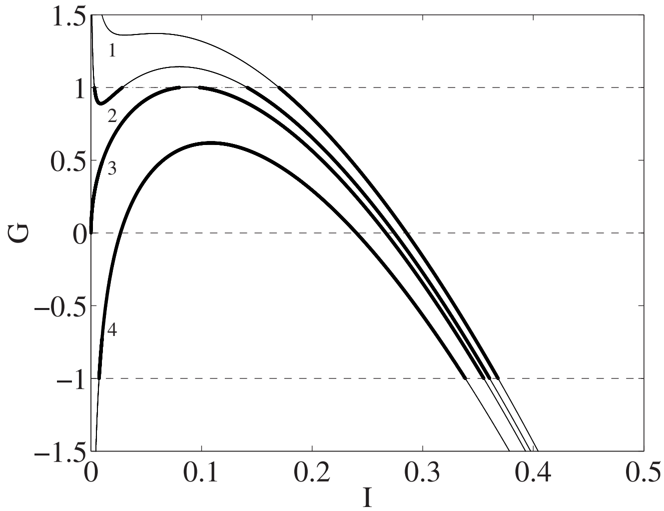

FIG. 3. The analysis of phase space trajectories for a single resonantly driven wave. The function $G=\sin\theta$ vs I for different energies h_i (indicated by numbers i in the figure): $h_1=-1\times10^{-3}$, $h_2=-4\times10^{-4}$, $h_3=0$, and $h_4=1.2\times10^{-3}$. The parameters are $a=0.0072$, $b=0.05$, and $c=3/16$. The allowed trajectories are shown by thick solid lines: there exists one open trajectory in the case of h_1, one closed and one open trajectory for h_2 and a single closed trajectory for h_4.

B. Classification of sister trajectories

Here, we discuss the sister trajectories in the constant parameters case. We express $\sin\theta$ from $H(I,\theta)=h$ and substitute it into Eq. (3) yielding

$$dI/dt = \pm\sqrt{P(I)}, \tag{5}$$

where

$$P(I) = 2a^2R - (h - bI + cI^2)^2. \tag{6}$$

The existence of sister trajectories is associated with a parameter regime, where the polynomial $P(I)$ has four real positive roots, I_1, I_2, I_3, and I_4. We will order the roots in this case as $I_1<I_2<I_3<I_4$, so the pairs $I_{1,2}$ and $I_{3,4}$ serve as the turning points [see Eq. (5)] of two sister trajectories, i.e., $I_1<I(t)<I_2$ for the lower I trajectory and $I_3<I(t)<I_4$ for the higher I trajectory. By rewriting $P(I)=c^2(I-I_1)(I_2-I)(I_3-I)\times(I_4-I)$ and substituting it into Eq. (5) we obtain

$$dI/dt = \pm c\sqrt{(I_4-I)(I_3-I)(I_2-I)(I-I_1)}. \tag{7}$$

Therefore, the periods of oscillations on the sister trajectories are

$$T_{1,2}(h) = \frac{2}{c}\int_{I_1}^{I_2}\frac{dI}{\sqrt{(I_4-I)(I_3-I)(I_2-I)(I-I_1)}}, \tag{8}$$

for the lower I trajectory, and

$$T_{3,4}(h) = \frac{2}{c}\int_{I_3}^{I_4}\frac{dI}{\sqrt{(I_4-I)(I_3-I)(I_2-I)(I-I_1)}}, \tag{9}$$

for the higher I trajectory. Then, in both cases,[26]

$$T_{1,2}(h) = T_{3,4}(h) = \frac{4}{c\sqrt{(I_4-I_2)(I_3-I_1)}}\mathbf{K}\left[\frac{(I_4-I_3)(I_2-I_1)}{(I_4-I_2)(I_3-I_1)}\right], \tag{10}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind. This equal period property $T_{1,2}(h)=T_{3,4}(h)\equiv T$ of the sister trajectories was first noticed in the case of a single driven wave,[1] but remains valid for two and three resonantly coupled waves.

A further understanding of the phase space structure associated with the sister trajectories can be obtained by using Eq. (2) to express

$$\sin\theta = \frac{bI - cI^2 - h}{a\sqrt{2R(I)}} \equiv G(I) \tag{11}$$

and studying the right hand side $G(I)$ of this expression versus different values of h. For example, in the case of a single driven wave ($R=I$), a typical dependence of $G(I)$ is shown in Fig. 3. Of physical interest are only the values of h such that there exist a region $I_{min}<I<I_{max}$, where $|G(I)|\leq 1$. We show these regions by thick lines in the figure. If $G(I_{min})=-G(I_{max})=\pm1$, the corresponding trajectory in phase space is open, otherwise the trajectory is closed. For each value of h there are at most four real turning points, I_1, I_2, I_3, and I_4, as discussed above. Consequently, there exist no more than two trajectories with the same value of h. In the case of a single driven wave, the function $G(I)$ has a single finite maximum for $h>0$ and, therefore, one encounters a pair of open sister trajectories if and only if this maximum is larger than unity. In contrast, for $h<0$, $G(I)$ may have or have not a finite maximum. If this maximum exists and exceeds unity, one may encounter one closed and one open sister trajectories. Furthermore, one can see in the figure that by starting from a closed sister trajectory by increasing the energy h,

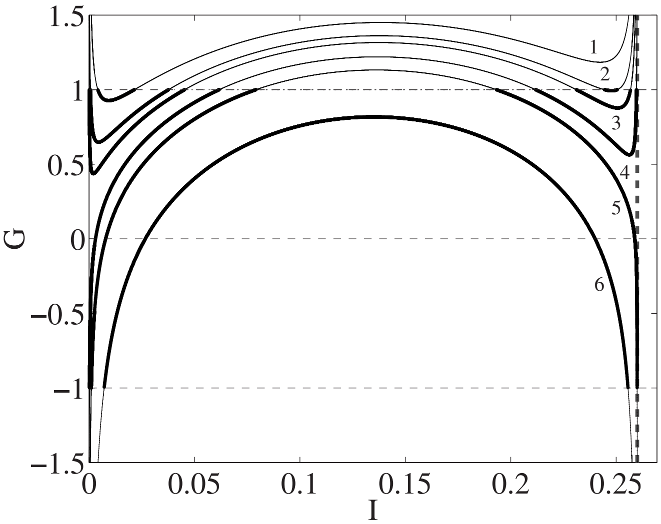

FIG. 4. The analysis of phase space trajectories for two resonantly coupled waves. The function $G = \sin\theta$ vs I for different energies h_i (indicated by numbers i in the figure): $h_1 = -4.5\times10^{-4}$, $h_2 = -2.2\times10^{-4}$, $h_3 = -1\times10^{-4}$, $h_4 = 1.5\times10^{-4}$, $h_5 = 3.8\times10^{-4}$, and $h_6 = 1.2\times10^{-3}$. The parameters are $a = 0.0142$, $b = 0.05$, $c = 3/16$, and $Q_1 = 0.26$. The allowed trajectories are shown by thick solid lines. In contrast with Fig. 1, one may have two closed sister trajectories (for h_2 and h_3) in the two coupled waves case.

one can reach the state where this closed sister trajectory becomes a fixed point. We shall use this property in the following.

Next, we discuss the sister trajectories in the case of two coupled waves $[R = I(Q_1 - I)]$. We show G versus I for different values of h in this case in Fig. 4. The form of these dependencies is similar to those in the single wave case (see Fig. 3) at small I, but differs at larger values of I, where there exists a possibility of *two closed* sister trajectories. This difference is easily understood by noting that G in the two wave case is obtained from that for a single driven wave by division by $(Q_1 - I)$ and, as the result, G may become $-\infty$ or $+\infty$ as $I \to Q_1$. It is the latter case, which yields the possibility of having the second closed sister trajectory (see Fig. 4). Importantly (see below), when starting from a pair of sister trajectories, one again can transform one of these trajectories to a fixed point by varying h. Finally, $G(I)$ in the case of three waves differs from that of a single driven wave by division by $(Q_1 - I)(Q_2 - I)$. Therefore, the general form of $G(I)$ in this case is similar to that in the two wave case and one again may have either one closed and one open or two closed sister trajectories (G may become $+\infty$ as $I \to Q_1$, assuming $Q_1 < Q_2$) and again, by varying h, one can reduce one of the sister trajectories to a fixed point.

III. ADIABATIC EVOLUTION AND PRESERVATION OF SISTER TRAJECTORIES

In this section we study the problem of preservation of sister trajectories under adiabatic variation of parameters. For systems with fixed parameters, the energy is a constant of motion, so two trajectories having the same energy initially will remain sisters at later times. In contrast, when parameters a, b, and c vary in time, the energies on initially sister trajectories generally become different functions of time and the trajectories cease to remain sisters. The goal is to find combinations of time varying parameters such that when starting with a pair of sister trajectories, the energies on the two trajectories remain the same. Our discussion of this issue involves the following three stages.

A. Difference between the action integrals

Let $J_{1,2}(h)$ and $J_{3,4}(h)$ be the adiabatically conserved action integrals associated with a pair of sister trajectories,

$$J = \frac{1}{2\pi}\oint I d\theta$$

(for open trajectories the definition of J involves integration between 0 and 2π). Since the period of the trajectory can be calculated via $T(h) = 2\pi\partial J/\partial h$,[27] we find that for the two sister trajectories

$$\partial J_{1,2}/\partial h = \partial J_{3,4}/\partial h, \tag{12}$$

i.e., the action difference $\Delta J \equiv J_{3,4}(h,a,b,c) - J_{1,2}(h,a,b,c)$ does not depend on h. This property will allow us to find ΔJ explicitly for any set of parameters a,b,c. With this goal in mind, we start on two given sister trajectories and vary h so that either (a) $J_{1,2}\to 0$, i.e., the lower trajectory becomes a fixed point, $I_1 = I_2 = I_l$, or (b) $J_{3,4}\to 0$, i.e., the higher trajectory becomes a fixed point, $I_3 = I_4 = I_h$. The second scenario is possible for two or three coupled wave systems only as discussed in Sec. II. Let us analyze these two cases separately:

(a) $I_1 = I_2 = I_l$, $J_{1,2}\to 0$. The variation in h does not change the value of ΔJ, but now ΔJ becomes

$$\Delta J = J_{3,4}(h,a,b,c). \tag{13}$$

At this stage, we adiabatically reduce a to zero (this adiabatic variation in a and the corresponding change in h comprises a *gedanken* procedure and is not related to the variation in h and a in our original problem). Then, by the adiabatic theory, $J_{1,2}$ remain zero, while the corresponding fixed point $I_l \to 0$ and the corresponding $h_l \to 0$. We may also vary b and/or c adiabatically as a slowly approaches zero and still $I_l \to 0$ and $h_l \to 0$. Suppose there exists a scheme (finding such a scheme is our final goal) enabling to vary the parameters a, b, and c so that the energy of the higher trajectory follows that of the lower trajectory $[h_{3,4}(t) \approx h_l(t)]$, i.e., $h_{3,4}(t)\to 0$ at the end of this adiabatic process. However at $a = 0$ and $h = 0$, $I_{3,4} = b/c|_{a=0}$ and, therefore,

$$\Delta J = J_{3,4} = \lim_{a\to 0}\frac{1}{2\pi}\oint I_{3,4}d\theta = \left.\frac{b}{c}\right|_{a=0}. \tag{14}$$

(b) $I_3 = I_4 = I_h$, $J_{3,4}\to 0$. As before, the variation in h does not change ΔJ, but now

$$\Delta J = -J_{1,2}(h,a,b,c). \tag{15}$$

At this point, we adiabatically reduce a to zero. Then, the corresponding fixed point becomes $I_h \to Q_1$. The same limit $I_h \to Q_1$ is obtained if b and/or c vary adiabatically in addition to $a \to 0$. Suppose again that, for

$a \to 0$, there exists a scheme of adiabatic variation b and c such that the energies on the two sister trajectories remain the same $h_{1,2}(t) \approx h_h(t)$ during this adiabatic process. Then, at the end, as $a=0$, $h_{1,2}=h_h \to bQ_1 - cQ_1^2$. However, for $h=bQ_1-cQ_1^2$ and $a=0$, one has $I_{1,2}=b/c|_{a=0}-Q_1$. Since $0<I<Q_1$ this situation is possible if, at the end of the process, $b/2c|_{a=0}<Q_1<b/c|_{a=0}$. Then, from Eq. (4), we find that in the final state, $d\theta/dt \neq 0$ and, therefore, the final trajectory is open, while

$$\Delta J = -J_{1,2} = -\lim_{a\to 0}\frac{1}{2\pi}\oint I_{1,2}d\theta = Q_1 - \left.\frac{b}{c}\right|_{a=0}. \quad (16)$$

Note that in both cases discussed above, the value of ΔJ depends on the final value $(b/c)|_{a=0}$ only in the aforementioned scheme of variation in parameters. Finding this variation scheme is the goal of Sec. III B.

B. Adiabatic variation scheme

Here we discuss the case when the trajectory reduced to a fixed point is the lower trajectory. The case of the higher fixed point can be treated similarly. Let us compare the energy increments $\Delta h_0 \equiv \Delta h_l$ and $\Delta h_{3,4}$ due to the adiabatic change of parameters after one oscillation period. Let $h_0 = h_{3,4}$ at some initial time. The initial oscillation periods on the two trajectories are the same [see Eq. (10)] and the trajectories remain sisters if $\Delta h_0 = \Delta h_{3,4}$ (i.e., $h_0 = h_{3,4}$) after one period of oscillations. Assuming adiabaticity and defining $D \equiv h_{3,4} - h_0$ and $\delta \equiv \Delta h_{3,4} - \Delta h_0$, we have

$$\begin{aligned}\delta &\approx \int_0^T \frac{dD}{dt}dt \\ &= \int_0^T\left(\frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt}\right)dt \\ &= \Delta_a\frac{da}{dt} + \Delta_b\frac{db}{dt} + \Delta_c\frac{dc}{dt},\end{aligned} \quad (17)$$

where

$$\Delta_q = 2\int_{I_{\min}}^{I_{\max}}\frac{\partial D}{\partial q}\frac{1}{dI/dt}dI, \quad q=a,b,c. \quad (18)$$

The last three integrals are calculated in the Appendix and the results are as follows:

$$\Delta_a = \frac{\pi}{a}(\sigma - 2b/c), \quad \Delta_b = \frac{2\pi}{c}, \quad \Delta_c = -\frac{\pi\sigma}{c}, \quad (19)$$

where $\sigma = 2I_0 + I_3 + I_4$ and $I_0 = I_{1,2}$ is the fixed point (the lower trajectory). Note that σ in the expressions for $\Delta_{a,c}$ is the sum of the roots of the polynomial (6) and, therefore, is equal to the coefficient at I^3 in the polynomial divided by c^2. Thus, $\sigma = 2b/c$ in the cases of a single driven or two coupled waves. Then, $\Delta_a = 0$, $\Delta_b = 2\pi/c$, $\Delta_c = -2\pi b/c^2$, and the difference between energy variations on the two sister trajectories during one oscillation period becomes [see Eq. (17)]

$$\delta = 2\pi\frac{d}{dt}\left(\frac{b}{c}\right). \quad (20)$$

Therefore, the trajectories remain sisters for constant ratio of b/c and any adiabatic variation in $a(t)$. In the case of a single driven wave, this is known[1] and we have now shown that the result holds for two coupled waves as well. In contrast, in the case of three coupled waves, $\sigma = 2b/c + 2a^2/c^2$ and, therefore, $\Delta_a = 2\pi a/c^2$, $\Delta_b = 2\pi/c$, and $\Delta_c = -2\pi(a^2/c^3 + b/c^2)$. This yields

$$\delta = \pi\frac{d}{dt}\left(\frac{a^2+2bc}{c^2}\right). \quad (21)$$

For example, in the three coupled waves case with $c=$const, one must have $a^2+2bc=$const to preserve the sister trajectories ($\delta=0$), as illustrated in our numerical example in Fig. 1.

Finally, we use these results for evaluating ΔJ in Eqs. (14) and (16). For this purpose, we must evaluate b/c at the limiting values of our variation scheme preserving the sister trajectories (i.e., when $a \to 0$). For example, in the cases of one or two wave systems, we must keep $b/c=$const. Then, for a single driven wave,

$$\Delta J = \frac{b}{c}, \quad (22)$$

and for the system of two coupled waves,

$$\Delta J = \begin{cases} \dfrac{b}{c}, & Q_1 > \dfrac{b}{c}, \\ Q_1 - \dfrac{b}{c}, & \dfrac{b}{2c} < Q_1 < \dfrac{b}{c}. \end{cases} \quad (23)$$

Note that the first line in Eq. (23) is a generalization of the previous result obtained for a single driven wave system.[1] Similarly, for three resonantly coupled waves, we must preserve $g \equiv (a^2+2bc)/c^2$, yielding

$$\Delta J = \begin{cases} \left.\dfrac{b}{c}\right|_{a=0} = \dfrac{g}{2}, & Q_1 > \dfrac{g}{2}, \\ Q_1 - \left.\dfrac{b}{c}\right|_{a=0} = Q_1 - \dfrac{g}{2}, & \dfrac{g}{4} < Q_1 < \dfrac{g}{2}. \end{cases} \quad (24)$$

C. Preservation of sister trajectories

At this stage we already evaluated the difference ΔJ between the action integrals for any pair of sister trajectories in a single driven, two-, and three-coupled wave systems. Our next and final goal is to show that the trajectories remain sisters (i.e., have the same energies continuously) for any adiabatic variation of parameters that preserves the value of ΔJ. We proceed from the differentials

$$dJ_i = \frac{\partial J_i}{\partial h_i}dh_i + \frac{\partial J_i}{\partial a}da + \frac{\partial J_i}{\partial b}db + \frac{\partial J_i}{\partial c}dc, \quad (25)$$

where $i=1,2(3,4)$ refers to the lower (higher) trajectory. For an adiabatic process, the actions J_i are adiabatic invariants ($dJ_i=0$) and one can use the last equation to express

$$dh_i = -\left(\frac{\partial J_i}{\partial a}da + \frac{\partial J_i}{\partial b}db + \frac{\partial J_i}{\partial c}dc\right)\left(\frac{\partial J_i}{\partial h_i}\right)^{-1}. \tag{26}$$

Taking the difference between the differentials dh_i of the two initially sister trajectories and using the property $\partial J_{3,4}/\partial h_{3,4} = \partial J_{1,2}/\partial h_{1,2} = 1/\Omega$ (Ω being the frequency of oscillations) yields

$$\begin{aligned} d(h_{3,4} - h_{1,2}) &= -\Omega\left(\frac{\partial(\Delta J)}{\partial a}da + \frac{\partial(\Delta J)}{\partial b}db + \frac{\partial(\Delta J)}{\partial c}dc\right) \\ &= -\Omega d(\Delta J). \end{aligned} \tag{27}$$

The last equation implies that any adiabatic variation in parameters preserving ΔJ [see Eqs. (22)–(24)] also leaves the energies of the trajectories equal, so the trajectories remain sisters. Figure 1 in Sec. II, where c and a^2+2bc are kept constant, illustrates this phenomenon in the case of R3WIs.

IV. CONCLUSIONS

In conclusion,

(a) We studied evolution of equal energy (sister) trajectories governed by a time dependent Hamiltonian of form (2) describing a single resonantly driven wave, as well as resonant two- and three-wave interactions in the presence of linear detuning and nonlinear frequency (wave vector) shifts. This type of Hamiltonians is characteristic of waves in either spatially uniform and time varying or 1D spatially varying, but time independent plasmas. The three aforementioned problems differ by the coupling function R in the Hamiltonian [$R=I$, $I(Q_1-I)$, and $I(Q_1-I)(Q_2-I)$ in the three cases, respectively]. The problem of the sister trajectories for a single driven wave case was studied previously,[1] while for two- and three-wave interactions this problem was analyzed here for the first time.

(b) We have shown that there may exist at most two sister trajectories in all the above-mentioned resonant interaction problems for a given set of parameters. Furthermore, the periods of oscillations characterizing these pairs of trajectories in all these problems are the same.

(c) We found conditions on slowly varying parameters in the problem, such that a pair of initially sister trajectories remain sister, i.e., their corresponding energies remain the same, despite variation in parameters. The condition for preservation of the sister trajectories in the case of two resonantly coupled waves (the MC problem) with slow parameters [see Eq. (20)] is the same as for the single resonantly driven wave, but for three-wave resonant interactions this condition is different [see Eq. (21)].

(d) Since the energies on the sister trajectories remain the same, our results allow to find the energy of a trajectory by following that of its sister trajectory under the above-mentioned variation of parameters without the need of numerical simulations.

(e) Possible applications of our theory could involve a three-wave resonant interaction, when one can neglect depletion of one of the pump waves. In this case, the problem reduces to that of MC with the undepleted pump amplitude serving as an externally controlled slowly varying parameter a in the problem. The phase mismatch b and nonlinearity coefficient c could remain constant for preservation of the sister trajectories. In addition, one may envision problems in optical systems (or plasmas), where the interacting waves propagate in coupled nonlinear waveguides (plasma channels), such that the nonlinearity and linear dispersion in different waveguides and the coupling can be controlled independently.

(f) There are restrictions on the initial conditions and parameters allowing preservation of sister trajectories, as shown in Fig. 1. In particular, given initial I and θ, which defines the energy h, the polynomial (6) must have four real and positive roots I_j initially. Furthermore, in systems of two and three interacting waves, all these roots must be less than $\min(Q_1, Q_2)$.

(g) We focused on the cases of stable resonant couplings. The problem of an unstable coupling (for two-wave interactions, for example, $R=I(Q_1+I)$ in such a case) can be treated similarly.

(h) Finally, it seems interesting to generalize our theory to four wave interactions, where one may have $R=I(Q_1+I)(Q_2-I)(Q_3-I)$ with constants Q_i associated with the three Manley–Rowe relations in this problem. The power of the characteristic polynomial (6) defining the turning points I_j for the sister trajectories is still four in this case and, again, one may have at most two sister trajectories. The case of five interacting waves seems to be qualitatively different.

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APPENDIX: INTEGRALS ASSOCIATED WITH ENERGY VARIATION DURING AN OSCILLATION

In this Appendix we calculate integrals

$$\Delta_q = 2\int_{I_{\min}}^{I_{\max}} \frac{\partial D}{\partial q}\frac{1}{dI/dt}dI, \quad q = a,b,c, \tag{A1}$$

where $D = h_{3,4} - h_0$ is the difference between the energies on two sister trajectories, such that one of the trajectories is the lower fixed point. We define

$$P_q = 2\int_{I_{\min}}^{I_{\max}} \frac{\partial h}{\partial q}\frac{1}{dI/dt}dI, \quad q = a,b,c$$

for each of the trajectories separately (so $\Delta_q = \Delta P_q$) and proceed to calculating

$$P_a = -2\int_{I_{\min}}^{I_{\max}} \frac{\sqrt{2R}\sin\theta}{dI/dt}dI. \tag{A2}$$

By using the definition, $h = bI - cI^2 - a\sqrt{2R}\sin\theta$, and Eq. (7) for dI/dt, the last integral can be written as

$$P_a = \frac{h}{a}S^{(0)} - \frac{b}{a}S^{(1)} + \frac{c}{a}S^{(2)}, \tag{A3}$$

where

$$S^{(n)} = \frac{2}{c}\int_{I_{\min}}^{I_{\max}} \frac{I^n dI}{\sqrt{(I_4-I)(I_3-I)(I_2-I)(I-I_1)}}. \tag{A4}$$

Similarly,

$$P_b = S^{(1)}, \quad P_c = -S^{(2)}. \tag{A5}$$

Then, taking the difference between P_q on the two sister trajectories, we obtain

$$\Delta_a = \frac{h}{a}\Delta S^{(0)} - \frac{b}{a}\Delta S^{(1)} + \frac{c}{a}\Delta S^{(2)} \tag{A6}$$

and

$$\Delta_b = \Delta S^{(1)}, \quad \Delta_c = -\Delta S^{(2)}, \tag{A7}$$

where $\Delta S^{(n)}$ is the difference between the values of $S^{(n)}$ on the trajectories.

Next, we evaluate the integrals $S^{(n)}$ for a pair of sister trajectories such that one trajectory is a fixed point, $I_{1,2}=I_0$, while the second trajectory is bounded by the two remaining turning points, $I_3<I<I_4$. We calculate these integrals by assuming $I_1 \approx I_2 \approx I_0$ and taking the limit $I_{1,2}\to I_0$. For example, in evaluating $S^{(n)}$ for the trajectory associated with the fixed point (we shall denote these integrals by $S_0^{(n)}$), we can approximate $(I_4-I)(I_3-I)$ by $(I_4-I_0)(I_3-I_0)$, while for the second sister trajectory (these integrals will be denoted by $S_{3,4}^{(n)}$), we approximate $(I-I_1)(I-I_2)$ by $(I-I_0)^2$. Consequently,

$$S_0^{(n)} = \frac{2I_0^n}{c\sqrt{(I_4-I_0)(I_3-I_0)}}\int_{I_1}^{I_2}\frac{1}{\sqrt{(I_2-I)(I-I_1)}}dI \tag{A8}$$

and

$$S_{3,4}^{(n)} = \frac{2}{c}\int_{I_3}^{I_4}\frac{I^n}{(I-I_0)\sqrt{(I_4-I)(I-I_3)}}dI. \tag{A9}$$

Then [see Eqs. (8)–(10)]

$$S_0^{(0)} = S_{3,4}^{(0)} = T, \tag{A10}$$

while

$$S_0^{(1)} = I_0 T, \quad S_0^{(2)} = I_0^2 T. \tag{A11}$$

Furthermore, by rewriting $S_{3,4}^{(1)}$ and $S_{3,4}^{(2)}$ in alternative forms, we reduce them to elementary integrals,

$$\begin{aligned} S_{3,4}^{(1)} &= \frac{2}{c}\int_{I_3}^{I_4}\frac{dI}{\sqrt{(I_4-I)(I-I_3)}} \\ &\quad + \frac{2}{c}\int_{I_3}^{I_4}\frac{I_0 dI}{(I-I_0)\sqrt{(I_4-I)(I-I_3)}} \\ &= \frac{2\pi}{c} + I_0 T, \end{aligned} \tag{A12}$$

$$\begin{aligned} S_{3,4}^{(2)} &= \frac{2}{c}\int_{I_3}^{I_4}\frac{IdI}{\sqrt{(I_4-I)(I-I_3)}} + \frac{2I_0}{c}\int_{I_3}^{I_4}\frac{dI}{\sqrt{(I_4-I)(I-I_3)}} \\ &\quad + \frac{2I_0^2}{c}\int_{I_3}^{I_4}\frac{dI}{(I-I_0)\sqrt{(I_4-I)(I-I_3)}} \\ &= \frac{\pi}{c}(I_3+I_4) + \frac{2\pi}{c}I_0 + I_0^2 T \\ &= \frac{\pi}{c}\sigma + I_0^2 T, \end{aligned} \tag{A13}$$

where $\sigma = I_3+I_4+2I_0$ is the sum of the four turning points I_i. Finally, these results for $S^{(n)}$ yield

$$\Delta S^{(0)} = 0, \quad \Delta S^{(1)} = \frac{2\pi}{c}, \quad \Delta S^{(2)} = \frac{\pi\sigma}{c}, \tag{A14}$$

and, in turn, using Eqs. (A6) and (A7), we find

$$\Delta_a = \frac{\pi}{a}(\sigma - 2b/c), \quad \Delta_b = \frac{2\pi}{c}, \quad \Delta_c = -\frac{\pi\sigma}{c}. \tag{A15}$$

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