Comparison of geometric and wave optics in an absorbing spherical plasma

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The absorption of laser radiation by a spherical plasma is considered via geometric optics. In addition to the rays, the amplitude of the electric field is found, and absorption by inverse bremsstrahlung is modeled. The authors find resonance absorption in a domain outside the region where geometric optics is valid, assuming there a locally one-dimensional plasma-density variation and matching the geometric-optics solution with the solution of the wave equation. The total absorption coefficient obtained from the ray-tracing method is compared with the numerical results based on an exact solution of Maxwell's equation, and excellent agreement is found.

I. INTRODUCTION

The absorption of laser light by plasmas has been studied extensively in recent years. In most of the theoretical and computational models the deposition of the absorbed energy in the plasma is found by solving the wave equation for the electromagnetic field inside the plasma, approximating the incident radiation as a plane wave.[1,2] Investigations have also been made of a laser beam focused on a spherical target.[3,4] The method was again based on the solution of the complete set of Maxwell's equations outside and inside the plasma, taking advantage of the spherical symmetry.

In this paper we present a different approach to the problem based on the geometric-optics approximation. The essence of the method is as follows: We describe the propagation of the electromagnetic waves in terms of rays, an approximate description valid when the local wavelength is much less than the local scale length. As the rays pass through the plasma, part of the electromagnetic energy is absorbed by the plasma. Since the geometric-optics approximation fails in the vicinity of the turning points of the rays, one must solve the wave equation in this region in order to proceed to the critical surface, where resonance absorption takes an additional fraction of the electromagnetic energy. If, however, the turning points and the critical surface are close enough, one can locally assume a linear variation of the plasma density. Then resonance absorption can be treated by matching the geometric-optics solution, as the ray approaches the turning point, with the well-known solution of the wave equation.

Although geometric optics has the advantage of applicability to a general plasma profile, provided the underlying inequality prevails, in the present work we apply it to the special case of a laser beam focused on a spherical plasma and compare our results with the numerical solutions of the wave equation.[3]

II. THEORY

Consider the wave equation for an inhomogeneous cold isotropic plasma, with plasma frequency[5] $\omega_p(\vec{r})$:

$$\nabla^2\vec{E} - \nabla(\nabla\cdot\vec{E}) + \{[\omega^2 - \omega_p^2(\vec{r})]/c^2\}\vec{E} = 0. \qquad (1)$$

In the geometric-optics approximation we seek a solution of Eq. (1) in the form

$$\vec{E}(\vec{r}) = \vec{a}(\vec{r})e^{i\psi(\vec{r})}, \qquad (2)$$

where if we define

$$\vec{k}(\vec{r}) = \nabla\psi(r), \qquad (3)$$

then $\delta = k|\nabla \ln\omega_p|^{-1} \ll 1$. If we also expand the amplitude $\vec{a} = \vec{a}_0 + \vec{a}_1 + \cdots$, where ascending terms are ordered in ascending powers of δ, then $\vec{a}_0$ obeys the equation

$$\{[c^2k^2 + \omega_p^2(\vec{r}) - \omega^2]\underline{I} - c^2\vec{k}\vec{k}\}\cdot\vec{a}_0(\vec{r}) = 0, \qquad (4)$$

which has a nonzero solution, namely a transverse wave ($\vec{a}_0 \perp \vec{k}$) solution, only if

$$\omega = [k^2c^2 + \omega_p^2(\vec{r})]^{1/2}. \qquad (5)$$

This is the well-known dispersion relation in an infinite homogeneous plasma with ω_p equal to the local plasma frequency at the point $\vec{r}$ in the inhomogeneous plasma considered here. The frequency ω in Eq. (5) plays the role of Hamiltonian in defining the equations of the rays[6]:

$$\dot{\vec{r}} = \omega_{\vec{k}} = (c^2/\omega)\vec{k}, \qquad (6)$$

$$\dot{\vec{k}} = -\omega_{\vec{r}} = -(1/2\omega)\nabla\omega_p^2(\vec{r}). \qquad (7)$$

We consider now the case of radiation incident on the spherical plasma as it is shown in Fig. 1. The center of the plasma is assumed to lie on the optical axis of the lens. We allow, however, the geometric focus of the lens to be at a certain distance s from this center. Since in our case $\vec{r}\times\nabla\omega_p^2=0$, the trajectories, similarly to those of the particles moving in a central field, lie in the plane of constant azimuth ϕ. If r and θ are polar coordinates in such a plane, then following (6) one gets

$$\dot{r}=(c^2/\omega)k_r. \qquad (8)$$

From conservation of the angular momentum we also have

$$r^2\dot{\theta}=(c^2/\omega)rk_\theta=cb=\text{const}, \qquad (9)$$

where b is the impact parameter of the ray. Then k_r in Eq. (8) is given by

$$k_r=\pm(k^2-k_\theta^2)^{1/2}=\pm\frac{\omega}{c}\left(1-\frac{b^2}{r^2}-\frac{\omega_p^2(r)}{\omega^2}\right)^{1/2}. \qquad (10)$$

The turning point r_0 of the ray is now defined by the equation

$$1-b^2/r_0{}^2-\omega_p^2(r_0)/\omega^2=0. \qquad (11)$$

The positive and negative signs in Eq. (10) correspond to incident and reflected waves, respectively. In order to avoid computational difficulties in integrating Eq. (8) in the vicinity of the turning points of the rays [difficulties arising as a result of a vanishing double-valued square root in Eq. (10)], we shall use a different equation for k_r, which can be derived from Eq. (7):

$$\dot{k}_r=k_\theta\dot{\theta}-\frac{1}{2\omega}\frac{d\omega_p^2(r)}{dr}, \qquad (12)$$

or using Eq. (9),

$$\dot{k}_r=\frac{\omega b^2}{r^3}-\frac{1}{2\omega}\frac{d\omega_p^2(r)}{dr}. \qquad (13)$$

Thus, finally, the rays in a spherical plasmas are given by the following set of equations:

$$\dot{r}=(c^2/\omega)k_r, \qquad (14)$$

$$\dot{k}_r=\frac{\omega b^2}{r^3}-\frac{1}{2\omega}\frac{d\omega_p^2}{dr}$$

$$\left[\text{or } k_r=\pm\frac{\omega}{c}\left(1-\frac{b^2}{r^2}-\frac{\omega_p^2(r)}{\omega^2}\right)^{1/2}\right], \qquad (15)$$

$$\dot{\theta}=bc/r^2. \qquad (16)$$

We proceed now to the transport of the amplitude of the electromagnetic field in the plasma. The general approach in solving this problem is based on the transport equation for the amplitude.[7] As shown in the Appendix, the electromagnetic field in our geometry can be divided into two indendent modes with the electric fields, respectively, in the plane of constant azimuth (TM mode) and perpendicular to this plane (TE mode). It is also shown that the amplitudes of the electric fields α_{TE} and α_{TM} of these modes satisfy the transport equation

$$\frac{d(\alpha^2)}{dt}=-\alpha^2\,\nabla\cdot\omega_{\vec{k}}-\nu\frac{\omega_p^2(r)}{\omega^2}\alpha^2,$$

$$\alpha=\alpha_{TE},\ \alpha_{TM}, \qquad (17)$$

where we have modeled absorption due to inverse bremsstrahlung by the electron-ion collision frequency ν. In order to solve this equation along the ray, one must also integrate an additional system of first-order differential equations which give the value $\nabla\cdot\omega_{\vec{k}}$ along the trajectory.[6] In our stationary case, however, Eq. (17) can be considerably simplified, transforming it into the transport equation for the energy flux in an infinitesimal flux tube containing the ray. Let us first rewrite Eq. (17) in the form

$$\nabla\cdot(\alpha^2\omega_{\vec{k}})=\nabla\cdot(\alpha^2\dot{\vec{r}})=-\nu(\omega_p^2/\omega^2)\alpha^2. \qquad (18)$$

Integrating this equation in the volume element $\delta V=\delta S\dot{r}\,dt$ of the infinitesimal flux tube (see Fig. 2) and using the Gauss theorem, we then have

$$\dot{J}=-\nu(\omega_p^2/\omega^2)J, \qquad (19)$$

where $J=\alpha^2\dot{r}\,\delta S$ is the flux of the electromagnetic energy in the flux tube. In order to take into account the resonance absorption of the TM mode,

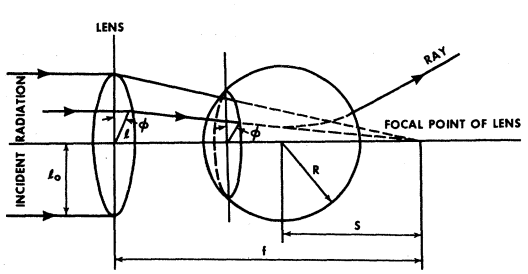

FIG. 1. Schematic diagram of laser radiation incident on a spherical plasma.

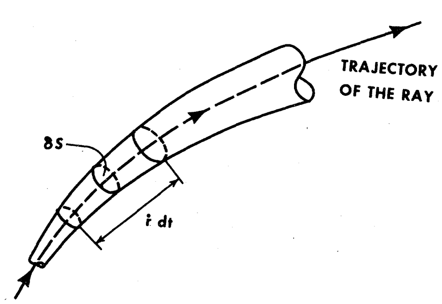

FIG. 2. Schematic of the infinitesimal flux tube containing the ray.

which is assumed to take place after the ray passes through the turning point, we can modify Eq. (19) and write it in the final form

$$\dot{J}_{\mathrm{TE}} = -\nu[\omega_p^2(r)/\omega^2]J_{\mathrm{TE}}\,, \tag{20}$$

$$\dot{J}_{\mathrm{TM}} = -\nu[\omega_p^2(r)/\omega^2]J_{\mathrm{TM}} - AJ_{\mathrm{TM}}(t-0)\delta(t-t_0)\,, \tag{21}$$

where t_0 is the time when the ray passes through the turning point and A is the coefficient of resonance absorption. The $-AJ_{\mathrm{TM}}(t-0)\delta(t-t_0)$ term in Eq. (21) represents the jump condition for J_{TM}, namely

$$J_{\mathrm{TM}}(t_0+0) - J_{\mathrm{TM}}(t_0-0) = -AJ_{\mathrm{TM}}(t_0-0)\,.$$

The coefficient A can be found from the following considerations: We assume first that the distance Δ between the turning points of the rays and the critical surface where the absorption takes place (see diagram in Fig. 3) is small enough so that $\Delta|\nabla \ln\omega_p^2| \ll 1$, and therefore one can assume a locally constant gradient of ω_p^2 in this region. Then, as is well known,[8] for the one-dimensional case with constant gradient $d(\omega_p^2/\omega^2)/dx = 1/d$, the absorption coefficient is determined by the parameter $\tau = (\omega d/c)^{2/3}D_1 \sin^2\beta_1$, where β_1 and D_1 are the angle of incidence and the dielectric constant $D = (ck/\omega)^2 = 1 - \omega_p^2(r)/\omega^2$ at the point r_1 the wave enters the region with linear plasma density profile. Since along the rays in this one-dimensional region the value of $D\sin^2\beta = (ck_y/\omega)^2$ is a constant defined by the y component of the wave vector in the incident wave, one can rewrite τ as

$$\tau = \tau_0 = (\omega d/c)^{2/3}D_0\,, \tag{22}$$

where D_0 is the dielectric constant at the turning point of the ray, where $\beta = \frac{1}{2}\pi$. Thus $A = A(\tau_0)$ depends *only* on the parameters of the turning point. This interpretation allows us to use the

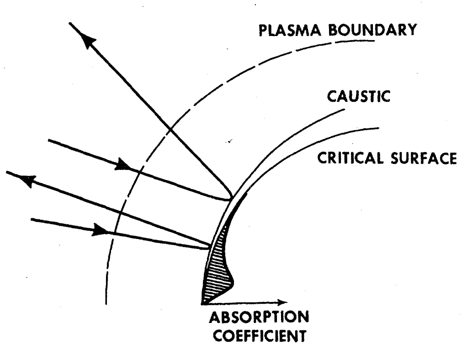

FIG. 3. Schematic of the plane of constant azimuth. The caustic is defined as an envelope of the turning points of the rays. The resonance absorption coefficient is shown schematically as a function of the position on the critical surface.

known one-dimensional expression for A in the general case, with τ_0 given by Eq. (22), where D_0 and $d = d_0$ are the dielectric constant and its gradient length at the turning point, respectively. These parameters are generally found by integrating the ray equations.

We use now the solution of Eqs. (20, 21)

$$J_{\mathrm{tot}}(t) = J_{\mathrm{TE}}(t) + J_{\mathrm{TM}}(t)$$

$$= \begin{cases} J_{\mathrm{tot}}^{\mathrm{in}} e^{-\alpha(t)}, & t < t_0, \\ (J_{\mathrm{tot}}^{\mathrm{in}} - AJ_{\mathrm{TM}}^{\mathrm{in}})e^{-\alpha(t)}, & t > t_0, \end{cases} \tag{23}$$

with $\alpha(t) = \int_0^t \nu(\omega_p^2/\omega^2)dt$, and find the distribution of the electric field in the incident and reflected waves. Suppose that the infinitesimal flux J_{tot} in Eq. (23) is defined by the element of area $\delta S_0 = l\,dl\,d\phi$ on the lens (see Fig. 1) and let $\delta S_r = r^2 \sin\theta\, d\theta\, d\phi$ be the surface element that the flux tube containing J_{tot} cuts on the sphere of radius r. Since the amplitude a_0 of the electric field in the wave is given by $a_0^2 \propto J_{\mathrm{tot}}/(\dot{r}\delta S_r)$, one has

$$\left(\frac{a_0(r)}{a_0^{\mathrm{in}}}\right)^2 = \frac{k_0}{k_r}\frac{1}{r^2}\frac{l}{\sin\theta}\frac{dl}{d\theta}\frac{J_{\mathrm{tot}}(r)}{J_{\mathrm{tot}}^{\mathrm{in}}}\,. \tag{24}$$

For example, the amplitude in the reflected wave outside the plasma, for the case of linear polarization of the incident beam, is given by

$$\left(\frac{a_0^{\mathrm{out}}}{a_0^{\mathrm{in}}}\right)^2 = \frac{1}{r^2}\frac{l}{\sin\theta}\frac{dl}{d\theta}(1 - A\cos^2\phi)e^{-\alpha(\infty)}. \tag{25}$$

Using this expression one can also find the total absorption coefficient

$$B = \int (a_0^{\mathrm{out}})^2 r^2 \sin\theta\, d\theta\, d\phi \Big/ \int (a_0^{\mathrm{in}})^2 l\, dl\, d\phi$$

$$= \int_0^{l_{\max}} (a_0^{\mathrm{in}})^2 (1 - \tfrac{1}{2}A)e^{-\alpha(\infty)} l\, dl \Big/ \int_0^{l_{\max}} (a_0^{\mathrm{in}})^2 l\, dl\,. \tag{26}$$

III. NUMERICAL RESULTS

In the following calculation our method was used for the case of a linearly polarized incident beam with intensity profile $J = J_0 \exp[-2(l/l_0)^5]$ and wavelength $\lambda = 1\ \mu$m. The beam was focused by the lens ($f\# = 0.5$) on the spherical plasma of radius $R = 150\ \mu$m. We assumed a linear dependence of the plasma density on radius $\omega_p^2(r)/\omega^2 = (R-r)/(R-r_c)$ and took the radius of the critical surface $r_c = 100\ \mu$m. In Fig. 4 the interval $\Delta = r_t - r_c$ between the turning points of the rays and the critical surface is shown for various values of the distances s between the geometric focus of the lens and the center of the plasma. It can be seen from these results that even at the perimeter of the laser beam ($l = l_0 = 0.5$ cm) one has

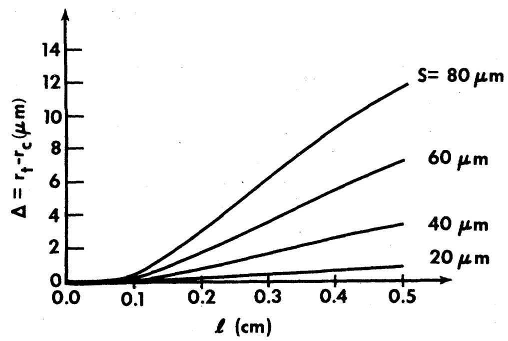

FIG. 4. Distance between the turning points of the rays and the critical surface vs position l in the incident laser beam.

$\Delta/r_c < 10\%$, and therefore the assumption of a one-dimensional locally linear variation of ω_p^2 in a thin layer in the vicinity of the turning and critical points (the assumption we make in finding the resonance absorption) is justified. Now one can solve Eqs. (14)–(16) for the trajectories of the rays for various l at the lens and find, for example, the geometric factor $(l/\sin\theta)dl/d\theta$, which, when Eq. (25) is used, gives the distribution of the intensity in the reflected wave. We present some of the results of these calculations in Fig. 5, where the dependence of

$$\Psi = \frac{J^{\text{in}}(l)}{J_0 l_0^2}\left[r^2\left(\frac{a_0^{\text{out}}}{a_0^{\text{in}}}\right)^2\right]\Bigg|_{r\to\infty}$$

on θ is shown for various values of ϕ and s. In these curves we assumed $\nu = 0$ $(T_e \to \infty)$ and used an analytic expression for $A = [1-(|Q|/|P|)^2]$, where the ratio $|Q|/|P|$ of the amplitude of the reflected and incident waves in the vicinity of the turning points was given by the approximate formula

$$\frac{|Q|}{|P|} = 1 - \frac{1}{1+1/(1.3\tau+0.7\tau^2)}\exp(-\tfrac{4}{3}\tau^{3/2}). \quad (27)$$

This expression is in good agreement with the results of the numerical calculations of the absorption coefficient[1] and gives the correct behavior $|Q|/|P| \simeq 1-1.3\tau$ at small τ (Ref. 9) and $|Q|/|P| \simeq 1-\exp(-\frac{4}{3}\tau^{3/2})$ at $\tau\to\infty$.[10] The fourth-fifth–order Runge-Kutta method was used in solving our system of first-order linear differential equations for the rays [Eqs. (14)–(16)]. The electron temperature effects on absorption via the inverse bremsstrahlung is taken into account in the results in Fig. 6, where the total absorption coefficient B [see Eq. (25)] is shown as a function of s for various temperatures (full line). The coefficient $I = \exp(-\alpha)$ in Eq. (26) was found by integrating the equation

$$\dot{I} = -\nu(\omega_p^2/\omega^2)I \quad (28)$$

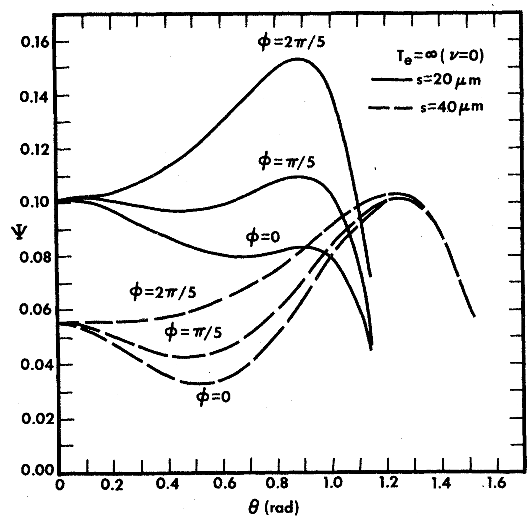

FIG. 5. Distribution of the electric field in the reflected wave outside the plasma.

[with I $(t=0)=1$] together with the equations of the rays. In Fig. 6 we also present (dotted line) the results of the numerical calculations based on the solution of the complete set of Maxwell's equations in the same geometry.[11] The good agreement between the two results illustrates the advantage of the proposed method, which exploits a simpler set of equations. An important feature of the ray-tracing method is demonstrated in Fig. 7, in which we make the same comparison as in Fig. 6 for the case when $r_c = 100$ μm and $R = 105$ μm. Very good agreement is seen even in this case, where the geometric-optics approximation might be expected to fail. The reason is that here the rays define the caustic of the turning points, or the boundary of the evanescent region, in which one must solve the wave equation in order to proceed to the critical surface, where the resonance absorption takes place. The absorption coefficient is thus found from the solution of the

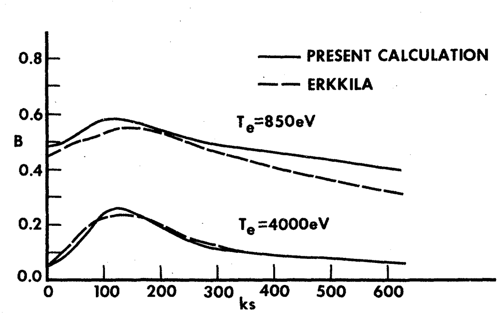

FIG. 6. Dependence of the total absorption coefficient on distance ks between the focal point of the lens and the center of the plasma, $r_c = 100$ μm, $R = 150$ μm.

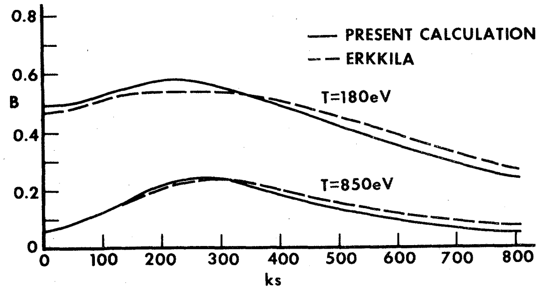

FIG. 7. Dependence of the total absorption coefficient on distance ks between the focal point of the lens and the center of the plasma; $r_c = 100\ \mu\text{m}$, $R = 105\ \mu\text{m}$.

wave equation by using the information provided by ray tracing.

ACKNOWLEDGMENT

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APPENDIX: REDUCTION OF THE TRANSPORT EQUATION IN AN ISOTROPIC PLASMA

Consider the dielectric and conductivity tensors in an isotropic plasma:

$$\underline{\epsilon} = [1 - (\omega_p^2 + c^2 k^2)/\omega^2]\underline{I} + (c^2/\omega^2)\vec{k}\vec{k}\,, \qquad \text{(A1)}$$

$$\underline{\sigma}^H = (\nu\omega_p^2/4\pi\omega^2)\underline{I}\,. \qquad \text{(A2)}$$

It is convenient to define the vectors

$$\vec{e}_3 = \vec{k}/k, \quad \vec{e}_2 = \vec{e}_\phi, \quad \vec{e}_1 = \vec{e}_2 \times \vec{e}_3\,, \qquad \text{(A3)}$$

and

$$\epsilon_1 = \epsilon_2 = 1 - (\omega_p^2 + c^2k^2)/\omega^2\,, \qquad \epsilon_3 = 1 - \omega_p^2/\omega^2\,. \qquad \text{(A4)}$$

Then (A2) can be written

$$\underline{\epsilon} = \epsilon_1 \vec{e}_1\vec{e}_1 + \epsilon_2\vec{e}_2\vec{e}_2 + \epsilon_3\vec{e}_3\vec{e}_3\,. \qquad \text{(A5)}$$

We are concerned now with those geometric-optics modes for which $\epsilon_1 = \epsilon_2 = 0$. The associated amplitude then has the form

$$\vec{a}_0 = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2\,. \qquad \text{(A6)}$$

The transport equations for the amplitudes α_j are given by [Eq. (38) in Ref. 6]:

$$\frac{d\alpha_j}{dt}\frac{\partial(\omega\epsilon_j)}{\partial\omega} + \beta_j = 0\,, \qquad \text{(A7)}$$

where

$$\beta_j = \sum_{l=1,2} \alpha_l \vec{e}_j^{\,*} \cdot \left\{ \frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\frac{\partial\vec{e}_l}{\partial t} + \frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\right)\right]\cdot\vec{e}_l - [(\nabla\vec{e}_l)^T\cdot\nabla_{\vec{k}}]\cdot\omega\underline{\epsilon}^T - \tfrac{1}{2}[\nabla\cdot(\nabla_{\vec{k}}\omega\underline{\epsilon})]\cdot\vec{e}_l + 4\pi\underline{\sigma}^H\cdot\vec{e}_l \right\}$$

$$= \tfrac{1}{2}\sum_l \alpha_l \left[\frac{\partial}{\partial t}\left(\vec{e}_j^{\,*}\cdot\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{e}_l\right) - \nabla\cdot[(\nabla_{\vec{k}}\omega\underline{\epsilon}):\vec{e}_l\vec{e}_j^{\,*}] + \frac{\partial(\omega\underline{\epsilon})}{\partial\omega}:\left(\frac{\partial\vec{e}_l}{\partial t}\vec{e}_j^{\,*} - \vec{e}_l\frac{\partial\vec{e}_j^{\,*}}{\partial t}\right) + 8\pi\underline{\sigma}^H:\vec{e}_l\vec{e}_j^{\,*} + (\nabla\vec{e}_j^{\,*})^T:(\nabla_{\vec{k}}\omega\underline{\epsilon})\cdot\vec{e}_l - (\nabla\vec{e}_l)^T:(\nabla_{\vec{k}}\omega\underline{\epsilon}^T)\cdot\vec{e}_j^{\,*}\right]. \qquad \text{(A8)}$$

However, since $\epsilon_l = 0$

$$\frac{\partial(\omega\underline{\epsilon})}{\partial\omega}\cdot\vec{e}_l = \frac{\partial}{\partial\omega}(\omega\underline{\epsilon}\cdot\vec{e}_l) - \omega\underline{\epsilon}\cdot\frac{\partial\vec{e}_l}{\partial\omega} = \frac{\partial}{\partial\omega}(\omega\epsilon_l\vec{e}_l) - \omega\underline{\epsilon}\cdot\frac{\partial\vec{e}_l}{\partial\omega} = \vec{e}_l\frac{\partial(\omega\epsilon_l)}{\partial\omega} - \omega\underline{\epsilon}\cdot\frac{\partial\vec{e}_l}{\partial\omega}\,, \qquad \text{(A9)}$$

and in a similar way one can show that

$$(\nabla_{\vec{k}}\omega\underline{\epsilon})\cdot\vec{e}_l = (\nabla_{\vec{k}}\omega\epsilon_l)\vec{e}_l - (\nabla_{\vec{k}}\vec{e}_l)\cdot\omega\underline{\epsilon}^T\,. \qquad \text{(A10)}$$

Thus

$$\beta_j = \tfrac{1}{2}\sum_l \alpha_l \left\{\delta_{jl}\left[\frac{\partial}{\partial t}\left(\frac{\partial(\omega\epsilon_j)}{\partial\omega}\right) - \nabla\cdot(\nabla_{\vec{k}}\omega\epsilon_j)\right] + 8\pi\underline{\sigma}^H:\vec{e}_l\vec{e}_j + \vec{e}_j^{\,*}\cdot\left(\frac{\partial(\omega\epsilon_l)}{\partial\omega}\frac{\partial\vec{e}_l}{\partial t} - (\nabla_{\vec{k}}\omega\epsilon_l)\cdot\nabla\vec{e}_l\right) - \vec{e}_l\cdot\left(\frac{\partial(\omega\epsilon_j)}{\partial\omega}\frac{\partial\vec{e}_j}{\partial t} - (\nabla_{\vec{k}}\omega\epsilon_j)\cdot\nabla\vec{e}_j^{\,*}\right) + \omega\underline{\epsilon}:\left(\frac{\partial\vec{e}_l}{\partial\omega}\frac{\partial\vec{e}_j^{\,*}}{\partial t} - \frac{\partial\vec{e}_l}{\partial t}\frac{\partial\vec{e}_j^{\,*}}{\partial\omega} - (\nabla_{\vec{k}}\vec{e}_l)^T\cdot\nabla\vec{e}_j^{\,*} + (\nabla\vec{e}_l)^T\cdot\nabla_{\vec{k}}\vec{e}_j^{\,*}\right)\right\}$$

$$= \tfrac{1}{2}\alpha_j\left(\frac{d}{dt}\frac{\partial(\omega\epsilon_j)}{\partial\omega} + \frac{\partial(\omega\epsilon_j)}{\partial\omega}\nabla\cdot\omega_{\vec{k}}\right) + \tfrac{1}{2}\sum_l \alpha_l\left[\frac{\partial(\omega\epsilon_j)}{\partial\omega}\left(\vec{e}_j^{\,*}\cdot\frac{d\vec{e}_l}{dt} - \vec{e}_l\cdot\frac{d\vec{e}_j^{\,*}}{dt}\right) + 8\pi\underline{\sigma}^H:\vec{e}_l\vec{e}_j^{\,*} + \omega\epsilon_3\vec{e}_3\vec{e}_3^{\,*}:\left(\frac{\partial\vec{e}_l}{\partial\omega}\frac{\partial\vec{e}_j^{\,*}}{\partial t} - \frac{\partial\vec{e}_l}{\partial t}\frac{\partial\vec{e}_j^{\,*}}{\partial\omega} - (\nabla_{\vec{k}}\vec{e}_l)^T\cdot\nabla\vec{e}_j^{\,*} + (\nabla\vec{e}_l)^T\cdot\nabla_{\vec{k}}\vec{e}_j^{\,*}\right)\right]. \qquad \text{(A11)}$$

For the case considered here, following Eqs. (A4) and (5)

$$\frac{\partial(\omega\epsilon_j)}{\partial\omega}=1+(\omega_p^2+c^2k^2)/\omega^2=2\,. \tag{A12}$$

Moreover, since the motion lies in a plane of constant azimuth

$$\frac{d\vec{e}_2}{dt}=0\,, \tag{A13}$$

and since the unit vectors are real and orthonormal

$$\frac{d}{dt}(\vec{e}_j^*\cdot\vec{e}_l)=\frac{d\vec{e}_j^*}{dt}\cdot\vec{e}_l+\vec{e}_j^*\cdot\frac{d\vec{e}_l}{dt}\,,\quad \frac{d\vec{e}_l}{dt}\cdot\vec{e}_j=0\,. \tag{A14}$$

In addition, since $\vec{e}_3=\vec{k}/k$, the last term in (A11) is proportional to

$$\vec{k}\vec{k}:[(\nabla\vec{e}_l)^T\cdot\nabla_{\vec{k}}\vec{e}_j-(\nabla_{\vec{k}}\vec{e}_l)^T\cdot(\nabla\vec{e}_j)]=[(\nabla\vec{e}_l)\cdot\vec{k}]\times[(\nabla_{\vec{k}}\vec{e}_j)\cdot\vec{k}]-[(\nabla_{\vec{k}}\vec{e}_l)\cdot\vec{k}]\cdot[(\nabla\vec{e}_j)\cdot\vec{k}]\,. \tag{A15}$$

But

$$(\nabla\vec{e}_l)\cdot\vec{k}=\nabla(\vec{e}_l\cdot\vec{k})-(\nabla\vec{k})\cdot\vec{e}_l=-(\nabla\vec{k})\cdot\vec{e}_l\,,$$
$$(\nabla_{\vec{k}}\vec{e}_j)\cdot\vec{k}=\nabla_{\vec{k}}(\vec{e}_j\cdot\vec{k})-(\nabla_{\vec{k}}\vec{k})\cdot\vec{e}_j=\vec{e}_j\,. \tag{A16}$$

Thus (A15) can be written

$$\vec{e}_j\cdot(\nabla\vec{k})\cdot\vec{e}_l-\vec{e}_l\cdot(\nabla\vec{k})\cdot\vec{e}_j=0 \tag{A17}$$

since $\nabla\vec{k}=\nabla\nabla\psi$ is a symmetric dyadic. Thus we can write Eq. (A7), using (A2),

$$2\frac{d\alpha_j}{dt}+\alpha_j\nabla\cdot\omega_{\vec{k}}+\nu\frac{\omega_p^2}{\omega^2}\alpha_j=0\,. \tag{A18}$$

[1]D. W. Forslund, J. M. Kindel, K. Lee, E. L. Lindman, and R. L. Morse, Phys. Rev. A 11, 679 (1975).
[2]K. G. Estabrook, E. J. Valeo, and W. L. Kruer, Phys. Fluids 18, 1151 (1975).
[3]J. Erkkila, thesis, University of California, 1974 (unpublished).
[4]J. J. Thomson, C. E. Max, J. Erkkila, and J. E. Tull, Phys. Rev. Lett. 37, 1052 (1976).
[5]V. L. Ginzberg, *Propagation of Electromagnetic Waves in Plasmas* (Pergamon, New York, 1970), p. 224.
[6]I. B. Bernstein, Phys. Fluids 18, 320 (1975).
[7]I. B. Bernstein and D. E. Baldwin, Phys. Fluids 20, 125 (1977).
[8]Ref. 5, Chap. IV.
[9]A. Ya. Omelchenko and K. N. Stepanov, Ukr. Fiz. Zh. 12, 1445 (1967) [Ukr. Phys. J. 12, 1396 (1967)].
[10]T. Tang, Radio Sci. 5, 111 (1970).
[11]Ref. 3, pp. 77–78.