General Geometric Optics Formalism in Plasmas

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Abstract—This paper exploits a general approach to geometric optics in inhomogeneous plasmas based on the properties of the local dielectric tensor $\bar{\bar{\epsilon}}$. We express $\bar{\bar{\epsilon}}$ in terms of its eigenvalues ϵ_j and eigenvectors $\hat{\bar{e}}$. Then to zeroth order in the geometric optics approximation the determinant $D = \epsilon_1\epsilon_2\epsilon_3$ vanishes and the elements ϵ_j vanish separately in pairs or simultaneously. It is shown that this branching in the dispersion relation changes the formulation of the geometric optics equations. The ray tracing and the transport of the amplitude of the wave in both degenerate and nondegenerate cases is described. The general procedure for transition through a boundary between degenerate and nondegenerate regions, where the rays split into two parts each following a different branch of the dispersion relation is also presented in this paper. We demonstrate our general method in a case, where radiation from a vacuum region enters an inhomogeneous magnetized plasma layer.

I. Introduction

THE GEOMETRIC optics approximation has been used extensively in studying propagation of radiation in inhomogeneous plasmas. This approach was first applied to the ionosphere where the basic assumption of geometric optics, the smallness of the local wavelength relative to the scale-length of variation of the plasma parameters, is satisfied even in 10^5-Hz frequency range. A description of these early works can be found in [1]. Recently geometric optics was successfully applied to problems in large laboratory plasmas [2]-[6], mostly in connection with electron cyclotron heating in various geometries, which became attractive with development of high-power microwave sources in ~1-cm wavelength range [7]. The geometric optics approximation was also used by the authors of this paper in studying the heating of laser produced microplasmas [8], where the basic assumption of geometric optics is satisfied in some cases since the wavelength of the incident laser radiation (usually in 1- to 10-μm wavelength range) is much less than the dimensions of the microplasma.

Thus the geometric optics has been shown to be an effective general approach in studying wave phenomena in a wide variety of plasmas and frequencies. The method, however, in most applications has not been completely exploited. Many studies were limited to ray tracing (finding the path of the energy flux), and almost no research has been done in studying the transport along the rays of the amplitude of the wave. Moreover, ray tracing usually has been performed by using special simple dispersion relations, each case treated separately; no attempt has been made to develop a general computer code, which would trace the rays and amplitudes using an arbitrary plasma dielectric tensor $\bar{\bar{\epsilon}}$, without reducing $\bar{\bar{\epsilon}}$ to some special form separately for each mode of interest. Some methods and ideas which can be used in creating such a general code will be discussed in this paper. The scope of the paper will be as follows: we will show in Section II that in general one must distinguish between ray tracing in a nondegenerate (anisotropic) plasma, where the branches of the dispersion relation are distinct (only one of the eigenvalues of $\bar{\bar{\epsilon}}$ vanishes) and in a degenerate plasma case, where two or three eigenvalues vanish simultaneously. The transport of the amplitudes in degenerate and nondegenerate plasmas will be discussed in Section III. Section IV is devoted to ray tracing through the boundary between degenerate and nondegenerate regions, where a ray splits into two parts, each following its own branch of the dispersion relation in the nondegenerate plasma. Finally, a numerical example, where the proposed general methods are used in the case of cold magnetized plasma, is given in Section V.

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II. Ray Equations

The study of a small signal electromagnetic wave propagation in a plasma conventionally proceeds from the Maxwell equations

$$\nabla \times \bar{B} = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}. \tag{1}$$

Assuming a linear response of the plasma to the electric field of the wave, one can write the current density $\bar{J}$ in (1) in the following general form [9] by using tensor conductivity kernal $\hat{\bar{\bar{\sigma}}}$

$$\bar{J}(\bar{r}, t) = \int d^3\bar{s} \int_0^\infty d\tau \hat{\bar{\bar{\sigma}}}(\bar{r}, t, \bar{s}, \tau) \cdot \bar{E}(\bar{r} - \bar{s}, t - \tau). \tag{2}$$

This form of $\bar{J}$ expresses casualty (dependence on the previous history of the plasma from $-\infty$ to t) and nonlocal properties of the plasma. If for simplicity one considers a stationary case, where $\bar{E}, \bar{B} \propto \exp(-i\omega t)$, (1) are reduced to the wave equation

$$-\frac{c^2}{\omega^2}\nabla \times \nabla \times \bar{E}(\bar{r}) + \bar{E}(\bar{r}) + \frac{4\pi i}{\omega}\int d^3\bar{s} \cdot \int_0^\infty d\tau\, e^{i\omega\tau}\, \hat{\bar{\bar{\sigma}}}(\bar{r}, \bar{s}, \tau) \cdot \bar{E}(\bar{r} - \bar{s}). \tag{3}$$

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In analogy with the case of infinite homogeneous plasma, where solutions of (3) can be represented by plane waves, in the case of a weak inhomogenity one can use perturbation analysis and seek a solution in the form

$$\bar{E}(\bar{r}) = \bar{a}(\bar{r}) \exp (i\psi(\bar{r})) \tag{4}$$

where if one defines

$$\bar{k}(\bar{r}) = \nabla\psi \tag{5}$$

it is assumed that the fractional change in $\hat{\bar{\bar{\sigma}}}$, when its argument $\bar{r}$ changes by $\lambda = 2\pi/k$, is small dimensionless parameter δ. In addition it is assumed that the fractional changes in $\bar{k}(\bar{r})$ and $\bar{a}(\bar{r})$ on distance λ are also of order δ. On using

$$\bar{a}(\bar{r}) = \bar{a}_0(\bar{r}) + \bar{a}_1(\bar{r}) + \cdots \tag{6}$$

where successive terms are ordered according to ascending powers of δ, inserting (6) in (3) and equating like powers of δ, one gets the following zeroth-order equation

$$\bar{\bar{\epsilon}} \cdot \bar{a}_0 = 0 \tag{7}$$

where the Hermitian tensor

$$\bar{\bar{\epsilon}}(\bar{k}, \bar{r}, \omega) = \left(1 - \frac{c^2k^2}{\omega^2}\right)\bar{\bar{I}} + \frac{c^2}{\omega^2}\bar{k}\bar{k} + \frac{4\pi i}{\omega}\bar{\bar{\sigma}}^A(\bar{k}, \bar{r}, \omega) \tag{8}$$

can be interpreted as the dielectric tensor in an infinite homogeneous plasma which has everywhere the properties the plasma under consideration displays at the point $\bar{r}$. The matrix $\bar{\bar{\sigma}}^A = \frac{1}{2}(\bar{\bar{\sigma}} - \bar{\bar{\sigma}}^{*T})$ is the anti-Hermitian part of the conductivity tensor

$$\bar{\bar{\sigma}} = \int d^3\bar{s} \int_0^\infty d\tau \hat{\bar{\bar{\sigma}}}(\bar{r}, \bar{s}, \tau)\, e^{i(\omega\tau - \bar{k}\cdot\bar{s})}. \tag{9}$$

We assume here that the Hermitian part $\bar{\bar{\sigma}}^H = \frac{1}{2}(\bar{\bar{\sigma}} + \bar{\bar{\sigma}}^{*T})$ of the conductivity tensor is operationally of order δ. It was demonstrated in [10] that formally when this assumption prevails, $\bar{\bar{\sigma}}^H$ can be taken into account in the first-order equation and then leads to weak growth or dissipation of the energy of the wave in the plasma.

Let us now express the Hermitian matrix $\bar{\bar{\epsilon}}$ in terms of its eigenvalues ϵ_j, $j = 1, 2, 3$ (which are real [11]) and eigenvectors $\hat{\bar{e}}_j$ (which are orthonormal and may be complex)

$$\bar{\bar{\epsilon}} = \epsilon_1\hat{\bar{e}}_1\hat{\bar{e}}_1^* + \epsilon_2\hat{\bar{e}}_2\hat{\bar{e}}_2^* + \epsilon_3\hat{\bar{e}}_3\hat{\bar{e}}_3^*. \tag{10}$$

Nontrivial solution of (7) exists only if the determinant of this matrix

$$D = \epsilon_1\epsilon_2\epsilon_3 \tag{11}$$

vanishes, giving in general three different branches (modes) of the dispersion relation

$$\omega^{(m)} = \omega^{(m)}(\bar{k}, \bar{r}), \quad m = 1, 2, 3. \tag{12}$$

When the dispersion relation for one of these modes, labeled by the mode n, is known in the form $\omega^{(n)} = \omega^{(n)}(\bar{k}, \bar{r})$, then $\omega^{(n)}$ can be used as the Hamiltonian of a quasi-particle having energy $\hbar\omega^{(n)}(\bar{k}, \bar{r})$ and momentum $\hbar\bar{k}$ at a point $\bar{r}$. Then the trajectories of such particles are described by the following Hamilton equations:

$$\dot{\bar{r}} = \frac{\partial\omega^{(n)}}{\partial\bar{k}}$$

$$\dot{\bar{k}} = -\frac{\partial\omega^{(n)}}{\partial\bar{r}}. \tag{13}$$

These equations are known as the ray equations, and clearly represent a short wavelength quasi-classical description of the electromagnetic field in the plasma wherein the wave can be interpreted as consisting of moving particles each carrying energy $\hbar\omega^{(n)}(\bar{k}, \bar{r})$, momentum $\hbar\bar{k}$, and having velocity equal to the group velocity $\partial\omega^{(n)}/\partial\bar{k}$ of the wave.

It is well known [12] that an explicit expression for $\omega^{(n)}$ in terms of $\bar{k}$ and $\bar{r}$ is not necessary in deriving the ray equations. For any implicit relation $H(\omega^{(n)}, \bar{k}, \bar{r}) = 0$, (13) can be rewritten in the form

$$\dot{\bar{r}} = -\frac{\partial H}{\partial\bar{k}}\Big/\frac{\partial H}{\partial\omega}$$

$$\dot{\bar{k}} = \frac{\partial H}{\partial\bar{r}}\Big/\frac{\partial H}{\partial\omega} \tag{14}$$

which can be easily verified by differentiating H with respect to $\bar{r}$ and $\bar{k}$ and considering $\omega^{(n)}$ in H as an implicit function of these variables.

The most general implicit dispersion relation is obtained by equating the determinant $D(\bar{k}, \bar{r}, \omega)$ of the matrix $\bar{\bar{\epsilon}}$ to zero. The possibility of branching in this dispersion relation, however, allows us to use D in (14) if only one of the eigenvalues ϵ_j $(j = 1, 2, 3)$ of $\bar{\bar{\epsilon}}$ vanish at the point of interest. Regions in the plasma with this property can be called *nondegenerate*. Inside the nondegenerate regions the convenient form of the ray equations is

$$\dot{\bar{r}} = -\frac{\partial D}{\partial\bar{k}}\Big/\frac{\partial D}{\partial\omega}$$

$$\dot{\bar{k}} = \frac{\partial D}{\partial\bar{r}}\Big/\frac{\partial D}{\partial\omega}. \tag{15}$$

These equations, however, cannot be used in *degenerate* plasmas, where two or three eigenvalues of $\bar{\bar{\epsilon}}$ vanish simultaneously. The reason is that in this case, according to (11), $\partial D/\partial\bar{k} = \partial D/\partial\bar{r} = \partial D/\partial\omega = 0$. Nevertheless, one can find another general Hamiltonian in a degenerate plasma by considering the characteristic equation for $\bar{\bar{\epsilon}}$

$$\det|\bar{\bar{\epsilon}} - \lambda\bar{\bar{I}}| = 0 \tag{16}$$

or

$$\lambda^3 - S\lambda^2 + F\lambda - D = 0 \tag{17}$$

where (see (10)) $S = \mathrm{Tr}\,(\bar{\bar{\epsilon}}) = \epsilon_1 + \epsilon_2 + \epsilon_3$ and $F = d_{11} + d_{22} + d_{33} = \epsilon_1\epsilon_3 + \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3$ is the sum of the second-order diagonal minors d_{ii} of $\bar{\bar{\epsilon}}$. Equation (17) contains all the information on the degeneracy of the plasma. It is now clear, for example, that in a doubly degenerate plasma both F and D vanish and thus F can be used in (14), defining nonsingular ray equations

$$\dot{\bar{r}} = -\frac{\partial F}{\partial \bar{k}} \bigg/ \frac{\partial F}{\partial \omega}$$

$$\dot{\bar{k}} = \frac{\partial F}{\partial \bar{r}} \bigg/ \frac{\partial F}{\partial \omega}. \tag{18}$$

Simplest examples of this situation occur for vacuum and for cold nonmagnetized plasmas. Both these cases are described by the conductivity tensor

$$\frac{4\pi i}{\omega} \bar{\bar{\sigma}}^A(\bar{r}) = -\frac{\omega_p^2(r)}{\omega^2} \bar{\bar{I}} \tag{19}$$

where $\omega_p(\bar{r}) = [(4\pi e^2/m) N_e(\bar{r})]^{1/2}$ is the electron plasma frequency at a point $\bar{r}$ of the plasma. The vacuum case corresponds to $\omega_p = 0$. Thus in a coordinate system with z axis along k vector ($k_x = k_y = 0$) the dielectric tensor (8) is diagonal with $\epsilon_1 = \epsilon_2 = 1 - (\omega_p^2 + c^2k^2/\omega^2)$ and $\epsilon_3 = 1 - (\omega_p^2/\omega^2)$. The wave solution is obtained only if $\epsilon_1 = \epsilon_2 = 0$, thus satisfying the definition of a doubly degenerate mode. It should be mentioned here that in general no diagonalization of $\bar{\bar{\epsilon}}$ is necessary in checking the degeneracy of a dispersion relation. One can simply calculate the sum F of the diagonal minors of $\bar{\bar{\epsilon}}$

$$F = \epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21} + \epsilon_{11}\epsilon_{33} - \epsilon_{13}\epsilon_{31} + \epsilon_{22}\epsilon_{33} - \epsilon_{23}\epsilon_{32} \tag{20}$$

which must be zero in the degenerate case.

Triply degenerate situation also can formally occur in the previous example. In fact $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1 - (\omega_p^2/\omega^2)$ at the points where $\bar{k} = 0$. In these regions, however, the wavelength becomes infinite and the main assumption of geometric optics is violated. A full wave analysis is necessary in such regions.

A less trivial case of a degenerate plasma region occurs in the presence of magnetic field. Here the conductivity tensor is given by

$$\frac{4\pi i}{\omega} \bar{\bar{\sigma}}^A = -i\omega_p^2 \begin{vmatrix} \dfrac{1}{\omega^2 - \omega_c^2} & -\dfrac{i\omega_c/\omega}{\omega^2 - \omega_c^2} & 0 \\ +\dfrac{i\omega_c/\omega}{\omega^2 - \omega_c^2} & \dfrac{1}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & \dfrac{1}{\omega^2} \end{vmatrix} \tag{21}$$

where $\omega_c = e|\bar{B}|/mc$ is the cyclotron frequency. In deriving a Hamiltonian for the ray equations in this case, it is convenient first to avoid numerical difficulties due to the resonance denominators in (21), by multiplying all the elements of $\bar{\bar{\epsilon}}$ by $\Delta = \omega - \omega_c$, which does not influence the dispersion relation. Now on using (20) it can be easily seen that the coefficient F has the following structure:

$$F = A\Delta + B\Delta^2 \tag{22}$$

and therefore F vanishes when $\omega_c(\bar{r}) = \omega$. Thus the plasma is degenerate on the cyclotron resonance surface. It is also clear that F changes its sign as one passes through the resonance; and, therefore, in contrast to the previous example of the plasma degenerated throughout a volume, one has a case of degenerate surface separating nondegenerate regions. The ray tracing through such degenerate surfaces desire a special treatment and will be discussed in Section IV.

III. Transport of Amplitudes

Partial information on the amplitude of the electric field of the wave can be obtained from the zeroth-order equation (7). In fact, expressing the amplitude in terms of eigenvectors $\hat{\bar{e}}_j$ of $\bar{\bar{\epsilon}}$

$$\bar{a}_0 = \alpha_1 \hat{\bar{e}}_1 + \alpha_2 \hat{\bar{e}}_2 + \alpha_3 \hat{\bar{e}}_3 \tag{23}$$

substituting this expression into (7) and using the orthonormality of the eigenvectors, one gets

$$\epsilon_1 \alpha_1 \hat{\bar{e}}_1 + \epsilon_2 \alpha_2 \hat{\bar{e}}_2 + \epsilon_3 \alpha_3 \hat{\bar{e}}_3 = 0. \tag{24}$$

Since the vectors $\hat{\bar{e}}_j$ are linearly independent, (24) can hold only if all the coefficients $\epsilon_j\alpha_j$ vanish. This implies that in a nondegenerate plasma, for example, when $\epsilon_1, \epsilon_2 \neq 0$ and $\epsilon_3 = 0$, one has $\alpha_1 = \alpha_2 = 0$ and therefore

$$\bar{a}_0 = \alpha_3 \hat{\bar{e}}_3. \tag{25}$$

In contrast, in the doubly degenerate case with $\epsilon_1 \neq 0$ and $\epsilon_2 = \epsilon_3 = 0$

$$\bar{a}_0 = \alpha_2 \hat{\bar{e}}_2 + \alpha_3 \hat{\bar{e}}_3. \tag{26}$$

Thus the zeroth-order equation gives us partial information about the polarization of the wave. The missing coefficients α_2 and α_3 in (25), (26) can be found from the first order in δ equations of the geometric optics approximation. This question was studied in [10] and here, for completeness, we present the final equations for the amplitudes:

$$\hat{\bar{e}}_l^* \cdot [\partial(\omega\bar{\bar{\epsilon}})/\partial\omega] \cdot \hat{\bar{e}}_l \frac{d\alpha_l}{dt}$$
$$= \sum_m \alpha_m \hat{\bar{e}}_m \cdot \left\{ \left[(\nabla \bar{e}_m)^T \cdot \frac{\partial}{\partial \bar{k}} \right] \cdot \omega \bar{\bar{\epsilon}}^T - \frac{1}{2} \left[\nabla \cdot \left(\frac{\partial}{\partial \bar{k}} \omega \bar{\bar{\epsilon}} \right) \right] \cdot \hat{\bar{e}}_m - 4\pi \bar{\bar{\sigma}}^H \cdot \hat{\bar{e}}_m \right\}. \tag{27}$$

The subscript l in this equation corresponds to the eigenvectors with zero eigenvalues, the time derivative is taken along the rays and $\nabla = \partial/\partial\bar{r} + (\nabla\bar{k}) \cdot (\partial/\partial\bar{k})$. The dyadic $\nabla\bar{k}$ itself is propagated along the ray by using equation

$$\frac{d(\nabla\bar{k})}{dt} = -\frac{\partial^2\omega}{\partial\bar{r}\partial\bar{r}} - (\nabla\bar{k}) \cdot \frac{\partial^2\omega}{\partial\bar{k}\partial\bar{r}} - \left[\frac{\partial^2\omega}{\partial\bar{r}\partial\bar{k}} + (\nabla\bar{k}) \cdot \frac{\partial^2\omega}{\partial\bar{k}\partial\bar{k}} \right] \cdot \nabla k \tag{28}$$

which is solved in parallel with the ray equations. Various second-order derivatives of ω in (28) can be obtained by implicit differentiation of the dispersion relation (the coefficients F or D according to the degeneracy of the plasma).

In order to use (27) in calculations one has to find eigenvectors corresponding to zero eigenvalues of $\bar{\bar{\epsilon}}$. In the nondegenerate case one can simply use the fact that since $F \neq 0$, at least one of the diagonal minors of $\bar{\bar{\epsilon}}$, say d_{jj}, is not equal to zero. Then, on using various minors d_{ij}, the components of the zero eigenvector are found from

$$e_{j_1} = A\frac{d_{jj_1}}{d_{jj}}, \quad e_{j_2} = A\frac{d_{jj_2}}{d_{jj}}, \quad e_j = A; \qquad j_1, j_2 \neq j, \; j_1 \neq j_2 \tag{29}$$

where A is the normalization constant.

In the degenerate case, when $\epsilon_2 = \epsilon_3 = 0$

$$\bar{\bar{\epsilon}} = \epsilon_1 \hat{e}_1 \hat{e}_1^* \tag{30}$$

where $\epsilon_1 = \mathrm{Tr}\,(\bar{\bar{\epsilon}})$. On multiplying (30) by a unit coordinate vector ($\hat{e}_x$, for example) one gets a simple algebraic equation for the nonzero eigenvector $\hat{e}_1$

$$\bar{\bar{\epsilon}} \cdot \hat{e}_x = \epsilon_1 \hat{e}_1 e_{1x}^*. \tag{31}$$

When this vector is found the zero eigenvectors can be constructed by using orthonormality of the eigenvectors. One of the possibilities is

$$\hat{e}_2 = \hat{e}_1^* \times \hat{e}_x; \qquad \hat{e}_3 = \hat{e}_1^* \times \hat{e}_2^*. \tag{32}$$

IV. Ray Tracing on Boundaries Between Degenerate and Nondegenerate Plasma Regions

Thus far we considered ray tracing and transport of the amplitudes in a purely degenerate or nondegenerate plasma. The transition across the boundary between such regions presents numerical difficulties and needs special treatment. In fact, if one starts in the degenerate region (for example, in vacuum and uses (18)), at the point the ray enters the nondegenerate plasma, one has to switch to the nondegenerate plasma rays equation (15). These equations, however, are singular on the boundary, where $D_{\bar{k}} = D_{\bar{r}} = D_\omega = 0$. The zero eigenvector at the boundary also cannot be found by using (29), since the determinants d_{ij} vanish. All these difficulties are not purely numerical. They arise since on the boundary between degenerate and nondegenerate plasma two coinciding branches of the dispersion relation separate, and the ray splits into two rays each following its own branch of the dispersion relation. We will now show that l'Hôpital's rule applied to (15), as one approaches the boundary from the nondegenerate side, gives in general two different values for $\bar{k}$ on the boundary, corresponding to different branches of the dispersion relation. In contrast, the group velocities $\dot{\bar{r}}$ of these modes on the boundary remain the same, as in the degenerate ray.

Consider a situation, where there exists a surface Q, dividing the plasma into degenerate and nondegenerate regions (see Fig. 1). Suppose that the ray starts from the degenerate region and splits into two parts at a time $t = t_0$ as it passes through the point 0 at the surface Q. If one goes backward in time along the rays 1 and 2 in the nondegenerate region, the numerators and denominators in (15) vanish at the boundary. If, however, the limits $\dot{\bar{r}}^+$ and $\dot{\bar{k}}^+$ for $\dot{\bar{r}}$ and $\dot{\bar{k}}$ exist, as one approaches the boundary from the nondegenerate side, they may be found by using l'Hôpital's rule

$$\dot{\bar{r}}^+ = -\frac{\lim\limits_{t\to t_0} \dfrac{dD_{\bar{k}}}{dt}}{\lim\limits_{t\to t_0} \dfrac{dD_\omega}{dt}} = -\frac{D_{\bar{k}\bar{k}} \cdot \dot{\bar{k}}^+ + D_{\bar{k}\bar{r}} \cdot \dot{\bar{r}}^+}{D_{\omega\bar{k}} \cdot \dot{\bar{k}}^+ + D_{\omega\bar{r}} \cdot \dot{\bar{r}}^+} \tag{33}$$

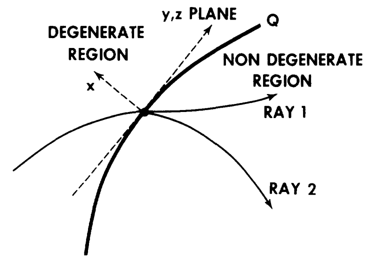

Fig. 1. Ray splitting on the boundary between degenerate and nondegenerate plasma regions.

and

$$\dot{\bar{k}}^+ = \frac{\lim\limits_{t\to t_0} \dfrac{dD_{\bar{r}}}{dt}}{\lim\limits_{t\to t_0} \dfrac{dD_\omega}{dt}} = \frac{D_{\bar{r}k} \cdot \dot{\bar{k}}^+ + D_{\bar{r}\bar{r}} \cdot \dot{\bar{r}}^+}{D_{\omega\bar{k}} \cdot \dot{\bar{k}}^+ + D_{\omega\bar{r}} \cdot \dot{\bar{r}}^+}. \tag{34}$$

Let us assume now that the eigenvalue ϵ_1 of $\bar{\bar{\epsilon}}$ does not vanish along the rays in Fig. 1. Then substituting the various derivatives of $D = \epsilon_1 \epsilon_2 \epsilon_3$ into (33) at the point 0, one gets

$$\dot{\bar{r}}^+ = -\frac{\epsilon_{2\bar{k}}(\epsilon_{3\bar{k}} \cdot \dot{\bar{k}}^+ + \epsilon_{3\bar{r}} \cdot \dot{\bar{r}}^+) + \epsilon_{3\bar{k}}(\epsilon_{2\bar{k}} \cdot \dot{\bar{k}}^+ + \epsilon_{2\bar{r}} \cdot \dot{\bar{r}}^+)}{\epsilon_{2\omega}(\epsilon_{3\bar{k}} \cdot \dot{\bar{k}}^+ + \epsilon_{3\bar{r}} \cdot \dot{\bar{r}}^+) + \epsilon_{3\omega}(\epsilon_{2\bar{k}} \cdot \dot{\bar{k}}^+ + \epsilon_{2\bar{r}} \cdot \dot{\bar{r}}^+)}. \tag{35}$$

If $\omega = \Omega(\bar{k}, \bar{r})$ is the dispersion relation for the degenerate mode, then because of the assumption of degeneracy one has

$$\begin{aligned} &\epsilon_2(\Omega(\bar{k}, \bar{r}), \bar{k}, \bar{r}) \equiv 0 \\ &\epsilon_3(\Omega(\bar{k}, \bar{r}), \bar{k}, \bar{r}) \equiv 0. \end{aligned} \tag{36}$$

Differentiation of these equations at the point 0 gives

$$\begin{aligned} \epsilon_{2\bar{k}} &= -\Omega_{\bar{k}} \epsilon_{2\omega} \\ \epsilon_{3\bar{k}} &= -\Omega_{\bar{k}} \epsilon_{3\omega}. \end{aligned} \tag{37}$$

Then (35) reduces to

$$\dot{\bar{r}}^+ = \Omega_{\bar{k}} = \dot{\bar{r}}^- \tag{38}$$

and, therefore, the group velocity remains unchanged as the ray crosses the boundary between degenerate and nondegenerate regions.

Similar considerations can be used in finding $\dot{\bar{k}}^+$. Suppose, for simplicity, that at the point 0 in Fig. 1 the x axis is normal to the surface Q. Then (36) still hold in the neighborhood of this point in z, y plane. Thus at the point 0 one has

$$\begin{aligned} \epsilon_{2y,z} &= -\Omega_{y,z} \epsilon_{2\omega} \\ \epsilon_{3y,z} &= -\Omega_{y,z} \epsilon_{3\omega} \end{aligned} \tag{39}$$

and similar to (38)

$$\dot{k}_y^+ = \dot{k}_y^-; \quad \dot{k}_z^+ = \dot{k}_z^-. \tag{40}$$

The unknown normal component $\dot{k}_x^+$ can now be found from (34), which is a quadratic equation, giving in general two different solutions for $\dot{k}_x^+$, corresponding to different rays in the nondegenerate region.

Thus by using the values $\dot{\bar{r}}^-$ and $\dot{\bar{k}}^-$, obtained by integration in the degenerate region, one already knows $\dot{\bar{r}}^+$ and can find two values of $\dot{\bar{k}}^+$ on the boundary, each giving rise to a new ray in the nondegenerate plasma. These limiting values of $\bar{k}$ and $\bar{r}$ can now be used to make a small time step δt from the boundary into the nondegenerate region by iterating, for example, the following equations:

$$y(t_0 + \delta t) = y(t_0) + \frac{\delta t}{2}\,[\dot{y}(t_0 + \delta t) + \dot{y}^+] \tag{41}$$

where y stands for $\bar{r}$ or $\bar{k}$ and $\dot{y}\,(t_0 + \delta t)$ is the right-hand side of (15). Starting from $t = t_0 + \delta t$, one can follow each of the two rays in the nondegenerate region by solving nonsingular equations (15).

Finally, the quantities $\dot{\bar{r}}^+$ and $\dot{\bar{k}}^+$ can be also used in finding two limiting zero eigenvectors of the nondegenerate $\bar{\bar{\epsilon}}$ matrix. This can be done again by applying l'Hôpital's rule in (29)

$$e^+_{j_1} = A\,\frac{(d_{jj_1})_{\bar{k}}\cdot\dot{\bar{k}}^+ + (d_{jj_1})_{\bar{r}}\cdot\dot{\bar{r}}^+}{(d_{jj})_{\bar{k}}\cdot\dot{\bar{k}}^+ + (d_{jj})_{\bar{r}}\cdot\dot{\bar{r}}^+}$$

$$e^+_{j_2} = A\,\frac{(d_{jj_2})_{\bar{k}}\cdot\dot{\bar{k}}^+ + (d_{jj_2})_{\bar{r}}\cdot\dot{\bar{r}}^+}{(d_{jj})_{\bar{k}}\cdot\dot{\bar{k}}^+ + (d_{jj})_{\bar{r}}\cdot\dot{\bar{r}}^+};\quad e_j = A. \tag{42}$$

We conclude by considering the problem of degenerate surfaces in a nondegenerate plasma, as, for example, the cyclotron resonance surface in the cold plasma model. In such a case l'Hôpital's rule can be applied in (15) from both sides of the degenerate surface, giving again

$$\dot{\bar{r}}^+ = \dot{\bar{r}}^-$$
$$\dot{k}^+_{y,z} = \dot{k}^-_{y,z}. \tag{43}$$

For the normal components $\dot{k}^+_x$ and $\dot{k}^-_x$ one gets the following quadratic equation (see (34))

$$\dot{k}^\pm_x = \frac{D_{x\bar{k}}\cdot\dot{\bar{k}}^\pm + D_{xr}\cdot\dot{\bar{r}}^\pm}{D_{\omega\bar{k}}\cdot\dot{\bar{k}}^\pm + D_{\omega\bar{r}}\cdot\dot{\bar{r}}^\pm}. \tag{44}$$

Therefore, if various second-order derivatives of D on the degenerate surface are continuous, (44) gives two solutions for each $\dot{k}^+_x$ and $\dot{k}^-_x$, and these solutions can be considered as corresponding to continuous rays

$$(\dot{k}^+_x)_{1,2} = (\dot{k}^-_x)_{1,2}. \tag{45}$$

Equations (43), (45) show that no new physical effects arise as various modes pass through the degenerate surfaces in a nondegenerate plasma and that the difficulties in integration of (15) are purely numerical, caused by the singularity of these equations at a certain point on the ray. The continuity of the rays can be used in avoiding these difficulties. For example, one can closely approach the degenerate surface by solving (15), then, by using the derivatives $\dot{\bar{k}}$ and $\dot{\bar{r}}$ at the last point of integration, make a small time step and jump over the pathological surface. After the jump the integration continues again by using (15). One can also use the values of $\dot{\bar{k}}$ and $\dot{\bar{r}}$ at a few points before the jump and get higher order prediction for values of $\bar{r}$ and $\bar{k}$ after the jump. Higher order methods allow one to start the jump earlier and make a larger time step, preserving the accuracy of the solution, thus avoiding integration in the close neighborhood of the degenerate surface.

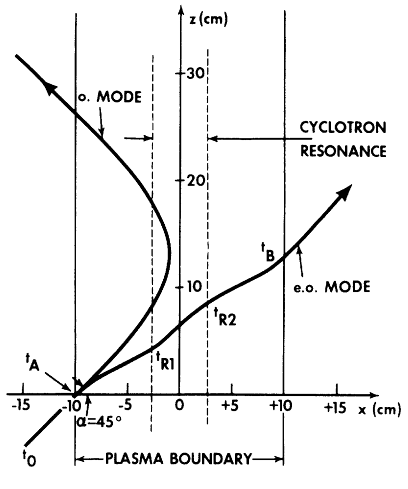

Fig. 2. Ordinary and extraordinary modes in the sample case.

IV. Numerical Example

In the following example we illustrate the application of the ideas of the previous sections to the case of radiation incident from vacuum (degenerate region) on a cold magnetized plasma layer. We consider, for simplicity, a traditional slab geometry with the magnetic field $B_x = B_y = 0$ and $B_z = B(x)$, and plasma density $n = n(x)$. We use the parabolic plasma density profile

$$n(x) = \begin{cases} 0, & |x| \geqslant a \\ n_0\left[1 - \left(\dfrac{x}{a}\right)^2\right], & |x| < a \end{cases} \tag{46}$$

with $n_0 = 1.1 \cdot 10^{13}\ \mathrm{cm}^{-3}$ and $a = 10$ cm. We also assume a magnetic mirror type of field with a minimum at $x = 0$, namely

$$B(x) = B_0\left[1 + \left(\frac{x}{l}\right)^2\right] \tag{47}$$

where $B_0 = 10^4$ and $l = 10$ cm. In Fig. 2 one can see the case, where the ray with a frequency of 30 GHz starts in vacuum at the point $x = -15$ cm, $y = 0$, $z = -5$ cm, at an angle of $\pi/4$ to the plasma layer. On the boundary with the plasma the ray splits into two parts (ordinary and extraordinary modes), each following a different trajectory in the plasma, crossing the cyclotron surface twice, and returning to the vacuum region. The algorithm used in this sample case is demonstrated in Fig. 3, where the parameters $A_F = F/(S^2 - 2F)$ and $A_D = 3D^{2/3}/(S^2 - 2F)$ are shown as a function of time along the ray, corresponding to the extraordinary mode. We use here normalized coefficients A_F and A_D rather than F and D, since they give a better measure of the degeneracy of the plasma. In fact, in terms of eigenvalues of $\bar{\bar{\epsilon}}$ one has

$$A_F = \frac{\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3}{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2} \qquad A_D = \left[\frac{(\epsilon_1\epsilon_2\epsilon_3)^{1/3}}{((\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)/3)^{1/2}}\right]^2 \tag{48}$$

and according to known algebraic inequalities $|A_F| \leqslant 1$ and $A_D \leqslant 1$. Also, it can be easily shown that if $A_D = \delta << 1$ and ϵ_1 is the eigenvalue of $\bar{\bar{\epsilon}}$ largest in magnitude, then for at least one of the remaining eigenvalues (say for ϵ_2) one has $|\epsilon_2/\epsilon_1| <$

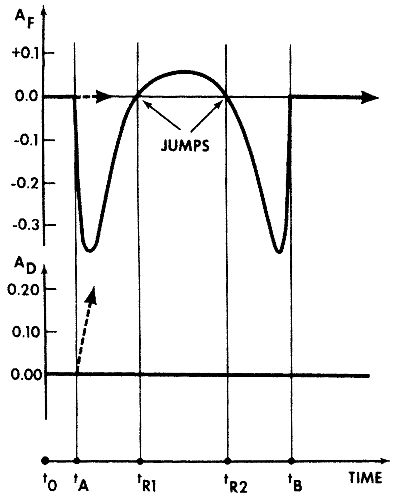

Fig. 3. The parameters A_F and A_D versus the time along the ray corresponding to the extraordinary mode.

$\delta^{3/4} << 1$. If one has $|A_F|, A_D < \delta << 1$, both $|\epsilon_2/\epsilon_1|$ and $|\epsilon_3/\epsilon_1|$ are less than $\delta^{3/4}$.

In the diagrams in Fig. 3 the ray starts in vacuum at $t = t_0$ and we integrate (18) for the degenerate case. Since along the ray

$$\dot{F} = F_{\bar{k}} \cdot \dot{\bar{k}} + F_{\bar{r}} \cdot \dot{\bar{r}} \tag{49}$$

the ray equations (18) force F and, therefore, A_F to remain zero in the degenerate region. The parameter A_D of course is also zero in this region. As one passes the boundary with the nondegenerate plasma at $t = t_A$ and continues to solve (18), the parameter A_F still remains zero. A_D, however, immediately grows in amplitude. This growth can be used as a diagnostic for the entry into a nondegenerate plasma region. It indicates that one is using the wrong ray equations and the calculation must switch at $t = t_A$ to (15) corresponding to the nondegenerate case. This false try of solving the degenerate plasma ray equations in the nondegenerate region is illustrated in Fig. 3 by the dotted lines. The sudden growth in A_D can be also used in the search of the boundary of the nondegenerate region. When the location of the boundary is known, one finds two solutions of (34) for $\dot{k}_x^+$ at the point t_A, takes one of these solutions (in our case corresponding to the extraordinary mode), makes a small step into the nondegenerate plasma, and then continues the ray by using (15). These equations force the parameter A_D to be zero along the ray. In contrast, A_F now grows in magnitude, reaches its maximum, and then decreases as one approaches the cyclotron resonance surface at $t = t_{R_1}$ and then vanishes. When A_F becomes small enough we make a jump across the resonance as it was described in the previous section. Then the integration continues; at $t = t_{R_2}$ the ray jumps a second time across the resonance surface and finally approaches the boundary of the plasma as A_F again diminishes. The location of the boundary is found and we switch again to the degenerate plasma ray equations (18) at $t = t_B$. This algorithm was applied on the DEC-20 computer. In the calculations the parameter A_D remained of the order of 10^{-5} along the rays in Fig. 1, which was a check of the satisfaction of the dispersion relation. The accuracy of the solution was also tested by solving the same problem using the Appleton-Hartree dispersion relation in the ray equations. In this case the ordinary and extraordinary modes are resolved and one has no numerical difficulties in integrating the ray equations through the boundary of the plasma and the cyclotron resonance surface. The comparison between the solutions by two methods shows that by using the general code one preserves an accuracy of at least four significant digits in the solutions for $\bar{r}$ and $\bar{k}$.

In conclusion, we have demonstrated the possibility of a general geometric optics code, which uses general information contained in the plasma dielectric tensor. It has also been shown that ray tracing and the transport of the amplitude is possible in degenerate and nondegenerate plasmas and on their boundaries without introducing a special dispersion relation for every mode of interest.

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