New perturbation expansion for anisotropic electron-velocity distribution functions in weakly ionized plasmas

L. Friedland and H. Eizenkiet
Center for Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, 91 904 Jerusalem, Israel
(Received 24 February 1987)

A novel perturbation method for calculating electron-energy distributions in weakly ionized plasmas with large inelastic collision cross sections is suggested. In contrast to the conventional two-term spherical-harmonics expansion, valid only in weakly anisotropic situations, the two-term expansion, developed here, is applicable uniformly for an arbitrary degree of anisotropy, thus describing both almost-isotropic and beamlike distribution functions. The procedure is basically an expansion of the integral form of the kinetic equation in powers of parameter $\delta=eE/mv\nu$ (E, v, and ν being the electric field, electron velocity, and total collision frequency) and is similar to that conventionally applied in WKB treatments of waves in weakly nonuniform environments. The predictions of the theory are compared with the results of an improved Monte Carlo simulation scheme, employing such advanced methods as Russian roulette, splitting, and "null" collisions.

I. INTRODUCTION

The classical problem of the evaluation of the electron-velocity distribution function $f(v,\theta)$ in weakly ionized plasmas in the presence of uniform electric field **E** is usually approached by solving the Boltzmann equation

$$-\frac{eE}{m}\left[\cos\theta\frac{\partial f}{\partial v}+\frac{\sin^2\theta}{v}\frac{\partial f}{\partial(\cos\theta)}\right]=\left[\frac{\partial f}{\partial t}\right]_c , \tag{1}$$

where θ describes the angle between the velocity vector **v** and the direction of the electron field and $(\partial f/\partial t)_c$ is the conventional collision term. The standard technique of simplifying this complex integro-differential equation is to introduce an approximate description of the θ dependence of f. Commonly, f is expanded in spherical harmonics

$$f(v,\theta)=f_0(v)+\cos\theta f_1(v)+\tfrac{1}{2}(3\cos^2\theta-1)f_2(v)+\cdots . \tag{2}$$

Then, since the θ dependence of the distribution is caused by the presence of the electric field, it is usually argued that for weak enough fields, we can truncate expansion (2), leaving only a limited number of terms. Thus (2) is viewed as an expansion of f in powers of a small parameter, proportional to E, so that $f_1\sim E$, $f_2\sim E^2$, etc.

The most widely used is the simplest, two-term spherical-harmonics expansion[1]

$$f\cong f_0+f_1\cos\theta . \tag{3}$$

Although successful in many applications, this approximation fails in some cases even for very weak electric fields. This situation is characteristic of gases in which the cross section for inelastic collisions in some electron-energy region becomes comparable to that of the elastic collisions.[2] In order to demonstrate the problem, we refer to the well-known results of the two-term spherical-harmonics expansion,[3]

$$f_1=-\frac{eE}{m}\frac{1}{\nu_m(v)}\frac{df_0}{dv} \tag{4}$$

and

$$\frac{e^2E^2}{3m^2}\frac{1}{v}\frac{d}{dv}\left[\frac{v^2}{\nu_m(v)}\frac{df_0}{dv}\right]=\sum_k\left[vf_0(v)\nu_{rk}(v)-v_k'f_0(v_k')\nu_{rk}(v_k')\right] , \tag{5}$$

where $\nu_m=N\sigma_m v$ is the frequency characterizing elastic collisions (N and σ_m being the number density of the gas molecules and the corresponding cross section, respectively). The summation in (5) is carried out over all the inelastic processes with characteristic energy losses ξ_k and collision frequencies $\nu_{rk}=N\sigma_{rk}v$, and we have defined v_k' via $\frac{1}{2}m(v_k')^2=\frac{1}{2}mv^2+\xi_k$. Equation (4) exhibits the intuitively expected dependence $f_1\sim E$. Nevertheless, Eq. (5) predicts a rapid decrease of f_0 with v in energy regions with relatively large inelastic cross sections and, as a result, f_1 may become large. Indeed, if for example ν_m and $\nu_r=\sum_k\nu_{rk}$ are constants and ν_r/ν_m is large enough, then $f(v_k')/f(v)\ll 1$ and we can neglect the last terms in the square brackets on the right-hand side of Eq. (5). Then this equation reduces to

$$\frac{e^2E^2}{3m^2\nu_m^2}\frac{d^2}{dv^2}(vf_0)\simeq\frac{\nu_r}{\nu_m}vf_0 , \tag{6}$$

yielding a decreasing solution of the form

$$vf_0\simeq\exp\left[-v\frac{\nu_m m}{eE}\left[\frac{3\nu_r}{\nu_m}\right]^{1/2}\right] . \tag{7}$$

Finally, the substitution of (7) into (4) gives

$$k=\frac{f_1}{f_0}=\left[\frac{3\nu_r}{\nu_m}\right]^{1/2}+\frac{eE}{m\nu_m v} . \tag{8}$$

Thus, we observe, that in the limit of weak fields, when

$$\delta=\frac{eE}{m\nu_m v}\ll\left[\frac{3\nu_r}{\nu_m}\right]^{1/2}, \tag{9}$$

the anisotropy factor $k=f_1/f_0$ does not depend on E at all. Moreover, already for $\nu_r/\nu\simeq 0.2$ (a typical value for vibrational excitations in N_2 and CO), k becomes of $O(1)$, thus formally invalidating the use of the two-term approximation in such cases.

The analysis which has led to Eq. (7), and the conclusions on the scaling of k with E in high-inelastic-loss regions, was still based on the results of the two-term spherical-harmonics expansion and several additional simplifying assumptions. Nevertheless, we shall show later, by using much fewer approximations, that for δ satisfying (9), the scaling

$$vf_0\simeq a\exp(-b/\delta), \tag{10}$$

where a and b are slowly varying functions of v (in a sense to be defined later), is characteristic of more general situations. This scaling was suggested in two unpublished works by Friedland[4] and Long.[5] The surprising observation of the independence of f_1/f_0 on E in high-energy-loss regions was also observed in computer simulations.[6]

Some results of these simulations are presented in Fig. 1. The figure shows the anisotropy factor k, in nitrogen, as a function of the electron energy for three values of E/N. We see that despite an order of magnitude difference in E/N, k does not depend on E/N in the energy range of 2–4 eV, where the gas is characterized by large vibrational excitation cross sections. Note also that the maximum value of 0.7 is in good agreement with predictions of Eq. (8) [for typical $(\nu_r/\nu_m)_{\max}\simeq 0.2$]. These large anisotropy factors show that the distribution in this energy range is quite beamlike [in the limiting case of a beam, when $f(v,\theta)\simeq g(v)\delta(\theta)/\sin\theta$, one has $k=3.0$] and we probably have reached the applicability limit of the two-term approximation.

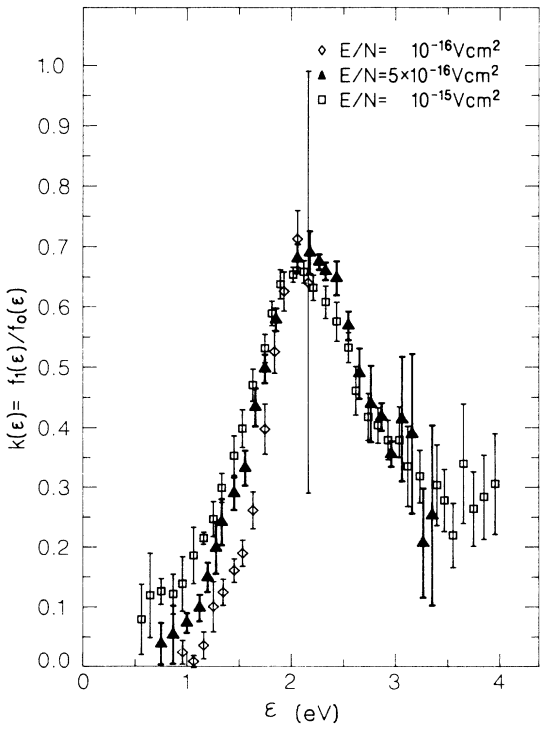

FIG. 1. The anisotropy factor $k=f_1/f_0$ in nitrogen.

In order to overcome the above-mentioned limitations of the two-term approach in the case of molecular gases characterized by large inelastic cross sections, a multiterm spherical-harmonics expansion method was developed recently.[7] The method has been systematically compared with the predictions of the two-term approach. It was demonstrated that, in practice, a relatively small number of terms (usually four or six) in Eq. (2) is sufficient to represent distributions found by Monte Carlo simulations[8,9] and Long's method.[5] Recently, the technique was extended to study effects of electrons produced in ionizations[10] and of the anisotropic scattering[11] on the distribution function. Thus, the multiterm expansion technique was shown to be a useful tool in studying anisotropic distribution functions in weakly ionized plasmas. Nevertheless, in limiting cases when the electrons are rather beamlike, the convergence of expansion (2) is slow, making the use of the multiterm spherical-harmonics expansion technique inconvenient.

In the present work we propose a new approach to the problem. The method is again an expansion in powers of a small nondimensional parameter δ [see Eq. (9)] proportional to E. Nevertheless, due to a proper treatment of the possibility of rapid variation of f_0 in the loss-dominated regions, the first *two* terms in this expansion can uniformly describe both swarm-type (weakly anisotropic) and beamlike situations. Our approach is a generalization of the theory of Ref. 6 and includes the possibility of slowly varying collision frequencies.

The presentation will be as follows. In Sec. II we will formulate our basic equation for the electron-energy distribution function. A perturbative solution of this equation will be developed in Sec. III. Finally, various examples of application of the theory will be considered in Sec. IV. We will also compare, in Sec. IV, our theory with the predictions of direct Monte Carlo simulations.

II. THE BASIC KINETIC EQUATION

We shall concentrate in this section on formulating the basic equation describing the electron-*energy* distribution function

$$F(\varepsilon)=\pi\sqrt{2}\left[\frac{e}{m}\right]^{3/2}\sqrt{\varepsilon}f_0\left[\frac{2e\varepsilon}{m}\right]^{1/2},$$

where $\varepsilon=mv^2/2e$ is the electron energy in electron volts and $f_0(v)$ is the first term in expansion (2). We assume that f_0 is normalized so that $\int_0^\infty F(\varepsilon)d\varepsilon=1$. Let us define now two auxiliary distribution functions $F^-(\varepsilon)$ and $F^+(\varepsilon)$, describing the electron-energy distributions in the gas just *before* and *after* a collision, respectively. These distribution functions have been introduced in Refs. 4 and 6 and very recently used by Ikuta and Murakami[12] in formulating and solving numerically an integral equivalent of the Boltzmann equation. A simple relation between F^-

and F exists, i.e.,[6]

$$F(\varepsilon)=F^{-}(\varepsilon)\frac{\langle\nu\rangle_{\rm av}}{\nu(\varepsilon)}\ , \tag{11}$$

where $\nu(\varepsilon)=N\sigma_t v$ is the total electron-collision frequency and $\langle v\rangle_{\rm av}=\int_0^\infty \nu(\varepsilon)F(\varepsilon)d\varepsilon$. On the other hand, since $F^{+}(\varepsilon)$ differs from $F^{-}(\varepsilon)$ due to *inelastic* collisions and $\nu_{rk}(\varepsilon)/\nu(\varepsilon)$ $(k=1,2,\ldots)$ can be viewed as a probability of an inelastic event of type k with the corresponding collision frequency $\nu_{rk}=N\sigma_{rk}v$, we can write

$$F^{+}(\varepsilon)=F^{-}(\varepsilon)-\sum_k\left[\frac{\nu_{rk}(\varepsilon)}{\nu(\varepsilon)}F^{-}(\varepsilon)-\frac{\nu_{rk}(\varepsilon+\xi_k)}{\nu(\varepsilon+\xi_k)}F^{-}(\varepsilon+\xi_k)\right]\ . \tag{12}$$

At this point, for simplicity, we will assume that the gas can be characterized by only one inelastic process with the characteristic energy loss ξ. In this case (12) becomes

$$F^{+}(\varepsilon)=F^{-}(\varepsilon)-\frac{\nu_r(\varepsilon)}{\nu(\varepsilon)}F^{-}(\varepsilon)+\frac{\nu_r(\varepsilon+\xi)}{\nu(\varepsilon+\xi)}F^{-}(\varepsilon+\xi)\ . \tag{13}$$

Note, now, that there exist two situations when Eq. (13) can be further simplified. The first is characteristic of rare inelastic collisions, when

$$\frac{\nu_r}{\nu}\ll 1\ . \tag{14}$$

Then, as a first approximation, (13) yields $F^{+}(\varepsilon)\simeq F^{-}(\varepsilon)$, which, after iterating again, gives

$$F^{+}(\varepsilon)\simeq F^{-}(\varepsilon)-F^{+}(\varepsilon)\frac{\nu_r(\varepsilon)}{\nu(\varepsilon)}+F^{+}(\varepsilon+\xi)\frac{\nu_r(\varepsilon+\xi)}{\nu(\varepsilon+\xi)}\ . \tag{15}$$

The second limiting situation occurs in the high-inelastic-loss energy regions, where

$$\frac{\nu_r}{\nu}\lesssim 1\ . \tag{16}$$

In this case, the distribution function F^{+} decreases rapidly with ε and we can assume that $F^{+}(\varepsilon+\xi)\ll F^{+}(\varepsilon)$. Then Eq. (13) yields

$$\begin{aligned}F^{+}(\varepsilon)&\simeq F^{-}(\varepsilon)-F^{-}(\varepsilon)\frac{\nu_r(\varepsilon)}{\nu(\varepsilon)}\\&=F^{-}(\varepsilon)-\frac{\nu_r(\varepsilon)/\nu(\varepsilon)}{1-\nu_r(\varepsilon)/\nu(\varepsilon)}F^{+}(\varepsilon)\\&\quad+\frac{\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}{1-\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}F^{+}(\varepsilon+\xi)\ .\end{aligned} \tag{17}$$

Thus, since cases (14) and (16) cover all but intermediate values of ν_r/ν, we will assume that for all values of ε we have approximately

$$F^{+}(\varepsilon)=F^{-}(\varepsilon)-Q(\varepsilon)\ , \tag{18}$$

where $Q(\varepsilon)$ is a function only of F^{+}, ν_r, and ν. Finally, since in a steady state, assumed here, the distribution F^{-} can be viewed as being formed from F^{+} due to the electron-energy changes *between* successive collisions, we can also write

$$\begin{aligned}F^{-}(\varepsilon)&=\int_0^\infty F^{+}(\varepsilon')\rho_{\varepsilon'}(\varepsilon-\varepsilon')d\varepsilon'\\&=\int_{-\infty}^{\varepsilon}F^{+}(\varepsilon-s)\rho_{\varepsilon-s}(s)ds\ ,\end{aligned} \tag{19}$$

where $\varepsilon-\varepsilon'=s$, and $\rho_\varepsilon(\Sigma)d\Sigma$ is the probability that the electron, having energy ε *just after* a collision, will have the energy in the interval $[\varepsilon+\Sigma,\varepsilon+\Sigma+d\Sigma]$ *just before* the next collision. Then Eq. (18) can be rewritten as

$$F^{+}(\varepsilon)=\int_{-\infty}^{\varepsilon}F^{+}(\varepsilon-s)\rho_{\varepsilon-s}(s)ds-Q(\varepsilon)\ , \tag{20}$$

which, if $\rho_\varepsilon(\Sigma)$ is known, can be viewed as an integral equation for $F^{+}(\varepsilon)$. Equation (20) is the desired integral kinetic equation for the electron-energy distribution function. Indeed, when $F^{+}(\varepsilon)$ is found from (20), we can compute $F^{-}(\varepsilon)$, by using Eq. (18), and finally, find $F(\varepsilon)$ via Eq. (11). This scheme was, in fact, used in numerical evaluations of the distribution function.[12] In the present paper we will further develop the theory and solve the problem perturbatively, focusing primarily on the energy regions characterized by large inelastic losses and therefore high anisotropy in the velocity distribution.

The probability $\rho_\varepsilon(\Sigma)$ cannot be obtained without specifying a concrete collision model. First, we will consider the most simple case of constant total collision frequency $\nu=$const. The results will be later generalized to include slow variation of ν with energy. For $\nu=$const we write

$$\Sigma=2\sqrt{\varepsilon B}su+Bu^2\ , \tag{21}$$

where $u=\nu t$, $s=\cos\theta$, t is the time between the collisions, θ represents the scattering angle, and

$$B=\frac{e^2E^2}{2m\nu^2}\ . \tag{22}$$

The differential distribution functions for the values of u and s, assuming the isotropic scattering model, are

$$\phi(u)=e^{-u} \tag{23}$$

and

$$\Psi(s)=\begin{cases}0, & |s|>1\\ \frac{1}{2}, & |s|\le 1\ .\end{cases} \tag{24}$$

Therefore, by definition, the probability of finding, independently, values of u and s in the intervals $(u,u+du)$ and $(s,s+ds)$, respectively, is $\phi(u)\,du\,\Psi(s)ds$. By transforming to new variables $u,s\rightarrow u,\Sigma$, where Σ is given by (21), we can write the probability of finding u and Σ in the intervals du and $d\Sigma$, respectively, as

$$\phi(u)\,du\,\Psi(s(u,\Sigma))\frac{\partial s(u,\Sigma)}{\partial\Sigma}d\Sigma\ ,$$

where [see (21)]

$$s(u,\Sigma)=\frac{\Sigma-Bu^2}{2\sqrt{\varepsilon B}\,u}\ . \tag{25}$$

Finally, by integrating over u we obtain the desired proba-

bility function

$$\rho_\varepsilon(\Sigma)d\Sigma=\int_{u=0}^{\infty}du\,\phi(u)\Psi(s(u,\Sigma))\frac{\partial s(u,\Sigma)}{\partial\Sigma}d\Sigma\ . \tag{26}$$

Thus, substituting (23) into (26), we obtain

$$\rho_\varepsilon(\Sigma)=\frac{1}{2\sqrt{B\varepsilon}}\int_0^\infty\frac{1}{u}e^{-u}\Psi(s(u,\Sigma))du\ . \tag{27}$$

Equation (27) completes the basic kinetic equation (20) for F^+ in the simplest case $\nu=$const. The next step is to develop an appropriate expansion scheme for solving Eq. (20).

III. PERTURBATION EXPANSION

At this point we observe that $\rho_{\varepsilon-s}(s)$ in Eq. (20) is a sharply peaked function of s, with a characteristic width Δ defined by the average electron-energy change between two successive collisions,

$$\Delta\simeq 2\sqrt{B\varepsilon}\ . \tag{28}$$

Consider a situation, when [see definition in Eq. (9)]

$$\delta=\frac{\Delta}{\varepsilon}=2\sqrt{B/\varepsilon}\ll 1\ . \tag{29}$$

In this case we can expand the weak energy dependence of $\rho_{\varepsilon-s}(s)$, associated with argument $\varepsilon-s$, in powers of s, and truncate the series in some low order. The assumption of this truncation procedure would be a relatively weak dependence of the electron-energy change between the collisions on the initial electron energy ε. This is a valid assumption when (29) is fulfilled. Nevertheless, we should be careful in attempting to expand $F^+(\varepsilon-s)$ in Eq. (20) in powers of s, even in this case.

Indeed, in the high-inelastic-loss regions, even for very low electric fields, when (29) is formally satisfied, F^+ can rapidly decrease with ε on a scale comparable with Δ (see the discussion in the Introduction). In such a case, the direct expansion of F^+ in powers of s, and the use of a truncated series in the integral, become invalid. In order to overcome this difficulty, we write

$$F^+(\varepsilon)=a(\varepsilon)e^{\Psi(\varepsilon)}\ , \tag{30}$$

where $a(\varepsilon)$ is assumed to be a slowly varying function of ε, in the sense that

$$\frac{1}{\alpha}\frac{d}{d\varepsilon}[\ln a(\varepsilon)]\sim O(\delta)\ ,$$

where $\alpha=\alpha(\varepsilon)$ is defined in the following. In contrast to $a(\varepsilon)$, function Ψ is allowed, to the lowest significant order, to vary as δ^{-1} [see Eq. (10)], but at the same time, if we define

$$\alpha(\varepsilon)=\frac{d\Psi(\varepsilon)}{d\varepsilon}\ , \tag{31}$$

then $\alpha(\varepsilon)$ is a weakly varying function of ε, i.e.,

$$\frac{1}{\alpha}\frac{d}{d\varepsilon}\ln\alpha\sim O(\delta)\ . \tag{32}$$

The arguments presented here are somewhat similar to those usually used in the conventional eikonal (WKB) expansion for wave propagation problems in a weakly inhomogeneous medium.

With the ordering just described, we can now use the truncated expansions separately for $a(\varepsilon)$ and $\Psi(\varepsilon)$ in (30), still taking into account the rapid variation of $F^+(\varepsilon)$. To the lowest order in δ, $F^-(\varepsilon)$ in Eq. (19) then becomes

$$F^-(\varepsilon)=\int_{-\infty}^{\varepsilon}\left[a(\varepsilon)-\frac{da}{d\varepsilon}s+\cdots\right]\exp\left[\Psi(\varepsilon)-\alpha(\varepsilon)s+\frac{1}{2}\frac{d\alpha(\varepsilon)}{d\varepsilon}s^2+\cdots\right]\rho_{\varepsilon-s}(s)\ . \tag{33}$$

Now we shall take into account the possibility of a slow variation of the total collision frequency ν with energy. In particular, we will assume that for a slowly varying $\nu(\varepsilon)[\alpha^{-1}d\ln\nu/d\varepsilon\sim O(\delta)]$, we can still use the expansion for $\rho_{\varepsilon-s}(s)$ derived for constant collision frequency ν_0, but will evaluate ν_0 in Eq. (20) at the average electron-energy value between the collisions, i.e., $\nu_0=\nu([(\varepsilon-s)+\varepsilon]/2)=\nu(\varepsilon-s/2)$. Then we can expand $\rho_{\varepsilon-s}(s)$ in Eq. (33) as

$$\rho_{\varepsilon-s}(s)=\rho^0_{\varepsilon-s}(s)-\frac{1}{2}\frac{\partial\rho^0_{\varepsilon-s}}{\partial\nu}\frac{d\nu}{d\varepsilon}s+\cdots\ , \tag{34}$$

where ρ^0 is given by Eq. (27) with $\nu=\nu(\varepsilon)$. Substitution of (34) into (33) yields

$$\begin{aligned}F^-(\varepsilon)&=e^{\Psi(\varepsilon)}\int_{-\infty}^{\infty}\left[\left[a-\frac{da}{d\varepsilon}s+\frac{1}{2}a\frac{d\alpha}{d\varepsilon}s^2\right]\rho^0_{\varepsilon-s}(s)-a\frac{s}{2}\frac{\partial}{\partial\nu}\rho^0_{\varepsilon-s}(s)\frac{d\nu}{d\varepsilon}\right]e^{-\alpha s}ds\\&=e^{\Psi}\left[aR+\frac{da}{d\varepsilon}\frac{\partial R}{\partial\alpha}+\frac{1}{2}a\frac{d\alpha}{d\varepsilon}\frac{\partial^2R}{\partial\alpha^2}-\frac{1}{2}\frac{\partial^2R}{\partial\nu\partial\alpha}\frac{d\nu}{d\varepsilon}a\right]\ ,\end{aligned} \tag{35}$$

where we have introduced

$$R(\alpha,\varepsilon)=\int_{-\infty}^{\infty}\rho^0_{\varepsilon-s}e^{-\alpha s}ds\ . \tag{36}$$

Finally, Eq. (20) reduces to

$$a(R-1)+\frac{da}{d\varepsilon}\frac{\partial R}{\partial\alpha}+\frac{1}{2}a\frac{d\alpha}{d\varepsilon}\frac{\partial^2R}{\partial\alpha^2}-\frac{1}{2}\frac{\partial^2R}{\partial\nu\partial\alpha}\frac{d\nu}{d\varepsilon}a=\frac{\nu_r(\varepsilon)/\nu(\varepsilon)}{1-\nu_r(\varepsilon)/\nu(\varepsilon)}a+\frac{\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}{1-\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}e^{\Psi(\varepsilon+\xi)-\Psi(\varepsilon)}a(\varepsilon+\xi)\ . \tag{37}$$

We shall show now that by solving Eq. (37) perturbatively, one obtains the distribution function in all cases of interest. The proper ordering of the terms in Eq. (37) can be done only when function $R(\alpha,\varepsilon)$ is known. This function is found in the Appendix, and the result is

$$R(\alpha,\varepsilon)=S+\frac{\alpha}{4\varepsilon}\frac{\partial^2 S}{\partial\alpha^2}\ , \tag{38}$$

where

$$S(\alpha,\varepsilon)=\sum_{n=0}^{\infty}\frac{(4B\varepsilon\alpha^2)^n}{2n+1}$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha')^{2n}}{2n+1}=\frac{1}{2\alpha'}\ln\left|\frac{1+\alpha'}{1-\alpha'}\right|=S(\alpha') \tag{39}$$

and

$$\alpha'=2\sqrt{B\varepsilon}\alpha\ . \tag{40}$$

We consider now the following two limiting cases.

For case (a),

$$\nu_r(\varepsilon)/\nu(\varepsilon)\sim O(\delta^3)\ , \tag{41}$$

i.e., when the inelastic losses are small and the right-hand side of Eq. (37) is of $O(\delta^3)$. In this case we view R as a series of powers of B ($\sim\delta^2$), namely,

$$R=1+\frac{4}{3}B\left[\frac{\alpha}{2}+\alpha^2\varepsilon\right]+O(B^2)\ . \tag{42}$$

Then to zero order in B, $R=1$ so that $\alpha(\varepsilon)\equiv 0$, and correct to the first order in B (second order in δ), Eq. (37) becomes

$$\tfrac{2}{3}B\left[\frac{da}{d\varepsilon}+\frac{1}{\nu}\frac{d\nu}{d\varepsilon}a\right]=\frac{\nu_r(\varepsilon)}{\nu(\varepsilon)}a-\frac{\nu_r(\varepsilon+\xi)}{\nu(\varepsilon+\xi)}a(\varepsilon+\xi)\ . \tag{43}$$

Here we have used $\partial R/\partial\alpha=2B/3$ and $\partial^2 R/\partial\alpha\partial\nu=-4B/3\nu$. Note that since $(1/\alpha\nu)d\nu/d\varepsilon$ is of $O(\delta)$ and the right-hand side of Eq. (43) is of $O(\delta^3)$, Eq. (43) describes the energy variation of a on the slow scale of $O(\delta)$, consistent with our perturbation procedure. We also observe that since $a(\varepsilon)\equiv F^+(\varepsilon)\simeq F(\varepsilon)$ ($\psi\equiv 0$ in this case), Eq. (43) is, in fact, the conventional equation (5) for the electron-energy distribution function. In addition, it follows from Eq. (8) that in case (a), in which (as was just shown) one can use the conventional two-term spherical-harmonics expansion method, the anisotropy factor $k\simeq eE/m\upsilon\nu=\delta$, indicating as expected the weak anisotropy.

For case (b),

$$\frac{\nu_r(\varepsilon)}{\nu(\varepsilon)}>O(\delta^2)\ . \tag{44}$$

In this case, Eq. (43) predicts rapid variation of a with energy, which contradicts our ordering assumptions and therefore the perturbation scheme used in case (a) is inapplicable. Nevertheless, we can correct the perturbation scheme by including the large right-hand side of Eq. (37) in the lowest order of the perturbation expansion. For example, consider the situation where the inelastic losses are considerable and we can view ν_r/ν as being of $O(1)$, so that to zero order, Eq. (37) reads

$$R-1=\frac{\nu_r(\varepsilon)/\nu(\varepsilon)}{1-\nu_r(\varepsilon)/\nu(\varepsilon)}$$
$$+\frac{\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}{1-\nu_r(\varepsilon+\xi)/\nu(\varepsilon+\xi)}\frac{a(\varepsilon+\xi)}{a(\varepsilon)}\frac{e^{\Psi(\varepsilon+\xi)}}{e^{\Psi(\varepsilon)}}\ . \tag{45}$$

Here, we will neglect the second term on the right-hand side, due to the rapid decrease of $\exp(\Psi)$ with energy, and assume that the resulting equation yields a solution for α which is of $O(\delta^{-1})$. Then $S\simeq O(1)$ and $(\alpha/4\varepsilon)(\partial^2 S/\partial\alpha^2)\simeq O(\delta)$, so that we can substitute $R\simeq S$ in (45), i.e.,

$$S(\alpha,\varepsilon)=S(\alpha')=1/(1-\nu_r/\nu)\ . \tag{46}$$

This equation defines α' [see Eqs. (39) and (40)], which being of $O(1)$ yields $\alpha=\alpha'/2\sqrt{B\varepsilon}$ of $O(\delta^{-1})$, as was assumed previously.

Next, with α known, we proceed to the first order in Eq. (37)

$$\frac{\alpha}{4\varepsilon}\frac{\partial^2 S}{\partial\alpha^2}a+\frac{da}{d\varepsilon}\frac{\partial R}{\partial\alpha}+\frac{a}{2}\frac{d\alpha}{d\varepsilon}\frac{\partial^2 R}{\partial\alpha^2}-\frac{1}{2}\frac{\partial^2 R}{\partial\nu\partial\alpha}\frac{d\nu}{d\varepsilon}a=0\ . \tag{47}$$

Here, we again can replace R by S and substitute

$$\frac{d\alpha}{d\varepsilon}=\frac{d}{d\varepsilon}\left[\frac{\alpha'}{2\sqrt{B\varepsilon}}\right]=-\frac{\alpha}{2\varepsilon}+\frac{d\alpha'}{d\varepsilon}\frac{\alpha}{\alpha'}+\frac{a}{\nu}\frac{d\nu}{d\varepsilon}\ .$$

This results in

$$\frac{da}{d\varepsilon}\frac{\partial S}{\partial\alpha}=\frac{a}{2}\left[\left[\frac{\partial^2 S}{\partial\alpha\partial\nu}-\frac{\alpha}{\nu}\frac{\partial^2 S}{\partial\alpha^2}\right]\frac{\partial\nu}{\partial\varepsilon}-\frac{\partial^2 S}{\partial\alpha^2}\frac{d\alpha'}{d\varepsilon}\frac{\alpha}{\alpha'}\right]\ , \tag{48}$$

which is the desired equation for the slowly varying function $a(\varepsilon)$ [note that the right-hand side is of $O(\delta)$].

We observe now that in the case $\nu,\nu_r=\text{const}$, Eq. (48) yields $a=\text{const}(\varepsilon)$ and the distribution F^+ becomes

$$F^+=a\exp\left[\int\alpha\,d\varepsilon\right]=ae^{\alpha'\sqrt{\varepsilon/B}}=ae^{2\alpha'/\delta}\ . \tag{49}$$

This result is consistent with our ordering assumption on the exponential factor $\exp(\Psi)$. It is interesting to note also that for $A=\nu_r/\nu$ small enough, Eq. (46) becomes

$$1+\frac{\alpha'^2}{3}+\cdots=\frac{1}{1-A}\simeq 1+A\ ,$$

so that

$$\alpha'\simeq d_0'=\pm[3A/(1-A)]^{1/2}\ . \tag{50}$$

Then, taking the negative solution (decreasing function F^+) for α', we find that Eq. (49) reduces to solution (7) for $f_0\upsilon\sim F(\varepsilon)\simeq F^+(\varepsilon)$, obtained by using the conventional two-term spherical-harmonics expansion. Thus, we conclude that the two-term expansion is valid only for $\nu_r/\nu\ll 1$.

For $\nu_r/\nu\sim O(1)$, the solution for the distribution function may significally depart from the one predicted by Eq. (7). Nevertheless, the form (49) of the solution still

preserves. In order to illustrate the difference between (49) and (7), we show in Fig. 2 the fractional departure of α' from the two-term spherical-harmonics solution α'_0 [see Eq. (50)] for different values of ν_r/ν. Note that since α' appears in the exponential, even a relatively small change in α' can influence the distribution function significantly. Thus, in conclusion, we presented a new expansion scheme which allows one to find the electron-energy distribution in weakly ionized plasmas in the presence of a *weak* electric field ($\delta = eE/m\nu V \ll 1$), in cases when the gas is characterized by slowly varying total collision frequency ($\alpha^{-1} d \ln\nu/d\varepsilon \ll 1$). In energy regions with small inelastic losses ($\nu_r/\nu \ll 1$) the new method yields results similar to those obtained by the conventional two-term spherical-harmonics expansion. In regions where the distribution is highly anisotropic due to large inelastic losses [$\nu_r/\nu \sim O(1)$] and the conventional expansion converges slowly, the new method still allows for the use of the two-term approximation.

IV. THE SIMULATION METHOD AND COMPARISON WITH THE THEORY

In this section we describe an improved Monte Carlo simulation scheme, useful especially in problems considered in this study. The approach is a combination of the "null"-collision method[13] with the "Russian roulette" and splitting methods.[14] To the best of our knowledge, this combination of methods is used here for the first time in the application to anisotropic distribution functions in gases in uniform electric fields.

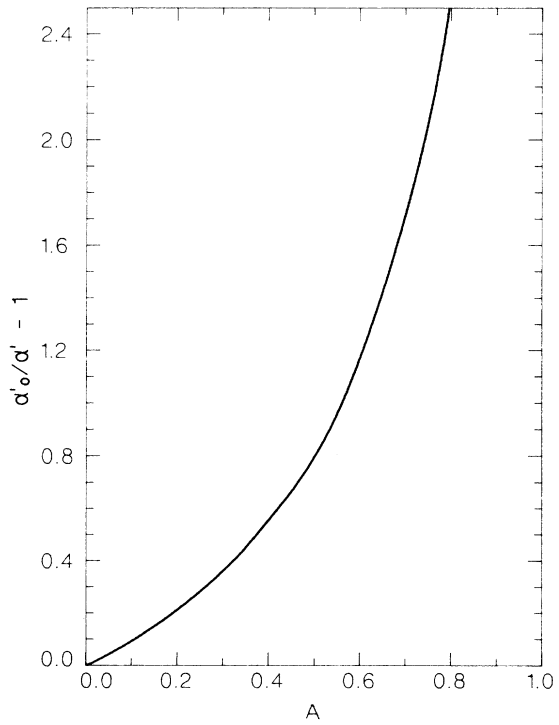

FIG. 2. The fractional departure of α' from the two-term spherical-harmonics solution α'_0 as a function of $A = \nu_r/\nu$.

The methods allow one to improve significantly two different aspects of the conventional Monte Carlo procedure. The null-collision method basically solves the problem of energy-dependent collision frequencies, characteristic of real scattering. It works as follows: One adds to the simulation an additional type of collision, i.e., the null collision, with the collision frequency $\nu_{\text{null}}(\varepsilon)$, such that the total collision frequency $\nu = \sum_i \nu_i(\varepsilon) + \nu_{\text{null}}(\varepsilon)$ is independent of energy. Then the time between the collisions is simulated by using a simple formula $t = -(1/\nu)\ln\gamma$, where γ are pseudorandom numbers, distributed uniformly in the interval [0,1]. In contrast to real collisions, resulting in an angle scattering and energy loss, the null collisions are assumed not to affect the electron trajectory during the "collision" process and, thus, do not influence the electron distribution function.

The splitting method is used to improve the statistics in the tail of the distribution, where usually the anisotropy is high, but the distribution function has small values. In the conventional Monte Carlo scheme the stationary distribution function is simulated by following a single test electron for a sufficient time. The probability of finding an electron in a given energy interval is proportional to the average time spent by the test electron in this interval. Since the distribution function is usually very small in the tail, the test electron spends relatively a very short time in this energy region, and the necessity to obtain sufficiently good statistics in the tail usually results in an unacceptable increase in the computing time. This feature of a poor statistics in the tail can be observed in Fig. 1, where the relative error in the value of $k = f_1/f_0$, at low values of E/N, approaches $\simeq 100\%$ in the tail, while the statistics at low energies is very good. The Russian roulette and splitting methods improve the statistics in the tail at the expense of the accuracy in the bulk of the distribution, which is usually high anyway. In the actual simulation, the energy axis is divided into intervals $\Delta_i = [\varepsilon_i, \varepsilon_{i+1}]$, $i = 0,1,2,\ldots$, where $\varepsilon_0 = 0$, and weight $M_i \geq 1$ is associated with each of the intervals. The lower the probability to find an electron in a particular interval, the higher the weight M_i that is associated with this interval. Assuming that $M_0 = 1$, we start the simulation procedure with a test electron in the interval Δ_0. The simulation is continued until the electron enters a new interval Δ_k. In this case, we replace the electron with a set of $N(k) = \prod_{i=1}^{k} M_i$ new "electrons," each having the weight $1/N$ and continue the simulation with a particular electron in this new set. If in the simulation process this electron enters a new interval Δ_{k_1}, with $k_1 > k$, the number of electrons is again increased and their weight decreased according to the weight of the new energy interval. If, on the contrary, the electron which starts in interval Δ_k reaches the interval Δ_{k_2}, with $k_2 < k_1$, it is "removed" from the simulation with the probability $1 - 1/N(k_2)$; otherwise, it is assumed to have weight $1/N(k_2)$, and the simulation is continued in the interval Δ_{k_2}. If the electron has been removed, the simulation procedure continues with a new test electron from the set created in interval Δ_k. Thus we create a *considerable number* of new electrons in the tail, where the distribution function is small, and the overall statistics is

thus improved. Since each electron in the tail has an appropriate small weight, the distribution itself is not affected by the described numerical electron fission-fusion procedure.

As a first application, we used our simulation scheme in the following test problem. We have assumed that $\nu=\text{const}(\varepsilon)$ and

$$\nu_r/\nu=\begin{cases}0, & \varepsilon\le\xi\\ A_0\left[1-\exp\left[-\dfrac{(\varepsilon-\xi)}{\varepsilon_0}\right]\right], & \varepsilon>\xi\ .\end{cases} \tag{51}$$

Figure 3 shows the distribution function $F^-(\varepsilon)$ for two sets of parameters (A): $\xi=4$ eV, $\varepsilon_0=10$ eV, $B=0.02$, and $A_0=0.6$ and (B): $\xi=4$ eV, $\varepsilon_0=10$ eV, $B=0.04$, and $A_0=0.8$. Note that function $F^-(\varepsilon)$ in the case of constant ν coincides with the conventional energy distribution function $F(\varepsilon)$ [see Eq. (11)]. The points in the figure represent the results of the simulation, while the solid lines have been obtained by solving Eq. (48) for $a(\varepsilon)$, substituting the result into expansion (30), and finally, transforming F^+ into F^- via Eq. (17). The error bars represent the statistical error of the Monte Carlo procedure, found by repeating the calculation 10 times, starting each time with an identical test electron. Every such simulation run involved 10^5 collisions, irrespective of whether the splitting had been introduced or not. The typical computing time was $\simeq 50$ sec CPU on the CDC Cyber 180-855 computer. The simulation included the above-mentioned splitting-roulette method, which could have been switched off, in order to test the advantages of the approach.

For the purpose of splitting, we divided the energy axis into splitting intervals of 2 eV each; and performed a trial simulation without the splitting first. This simulation was used for estimating the approximate ratios, $f(\varepsilon_i)/f(\varepsilon_j)$, of the distribution function in the ith and the jth energy intervals. These ratios allowed us to estimate the splitting weights M_k by using $M_j/M_i=f(\varepsilon_i)/f(\varepsilon_j)$ and $M_0=1$. Such a choice of M_k was probably not the optimal one, the optimization being outside the scope of the present work.

Figure 4 shows anisotropy factor $k=f_1/f_0$ in the test problem and demonstrates the dramatic statistical improvement one obtains by using the described splitting technique. The solid circles represent the results of the nonsplitted simulation. One can see that the simulation becomes meaningless at $\varepsilon>10$ eV, when the error exceeds 100%. The open circles, in contrast represent the use of the splitting technique and demonstrate a considerable increase in the accuracy (by using essentially the same computing time, since the total number of collisions remained unchanged). Note that anisotropy factor k in the test case becomes significant and reaches a constant ($\simeq 2$) at high energies, when $\nu_r\to$const.

Finally, in Fig. 5 we show the case when $\nu\neq$const. Particularly, we have used

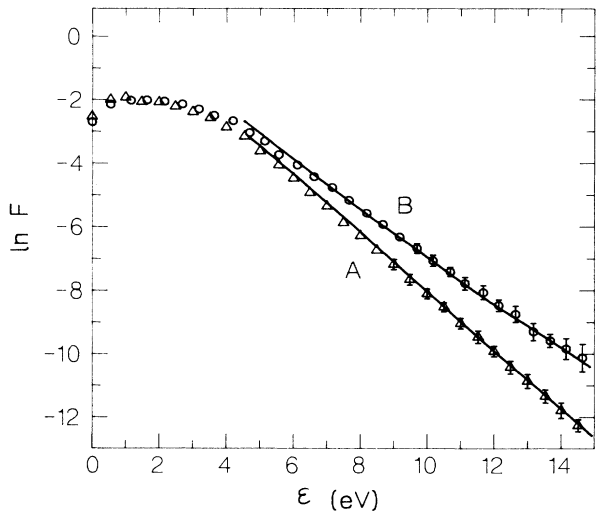

FIG. 3. The distribution function F^- for various values of A and constant total collision frequency ν. The parameters in the graph are A: $\xi=4$ eV, $\varepsilon_0=10$ eV, $B=0.02$, and $A_0=0.6$ and B: $\xi=4$ eV, $\varepsilon_0=10$ eV, $B=0.04$, and $A_0=0.8$. The points and the error bars represent the simulation results, while the solid lines show the theoretical prediction.

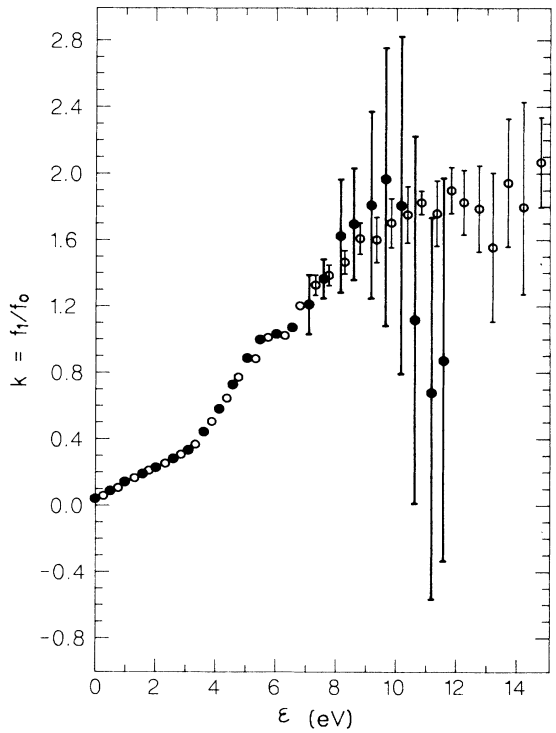

FIG. 4. The anisotropy factor $k=f_1/f_0$ vs energy in the test case. The parameters are $\xi=4$ eV, $\varepsilon_0=6$ eV, $B=0.02$, $A_0=0.9$, and $\nu=$const. The solid circles represent the results of the simulations without the electron splitting. The open circles correspond to the simulations with the splitting and Russian roulette methods included.

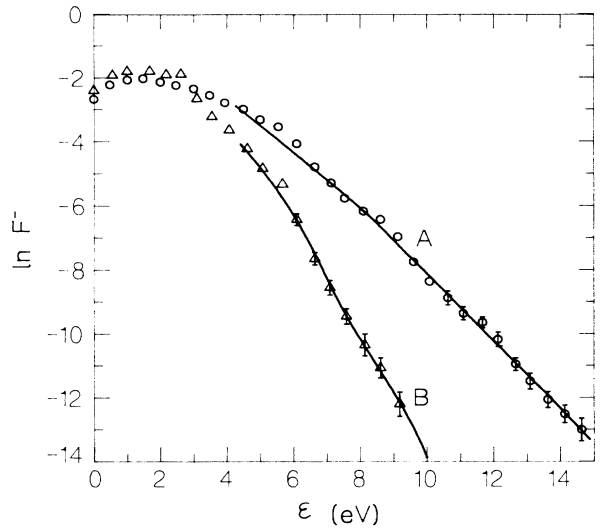

FIG. 5. The distribution function F^- in the model with energy-dependent total collision frequency $\nu(\varepsilon)$ [see the definitions in Eq. (52)]. The two cases shown correspond to A: $B\nu^2=0.01$ and B: $B\nu^2=0.002$. The points with error bars represent the simulation results, while the solid lines are the results of the new perturbation expansion scheme developed here.

$$\nu_{\rm el}=0.6\exp[-\varepsilon({\rm eV})/30]\ ,$$
$$\nu_r=\begin{cases}0, & \varepsilon\le 30\ {\rm eV}\\ 1-\exp\left[\dfrac{4-\varepsilon({\rm eV})}{30}\right], & \varepsilon>30\ {\rm eV}\ .\end{cases} \tag{52}$$

The examples in the figure correspond to the cases $B\nu^2=0.01$ [graph (A)] and $B\nu^2=0.002$ [graph (B)]. The points and solid lines, as before, represent the simulation and the theory, respectively. An excellent agreement can again be seen in the figure. Thus, we conclude that our perturbation expansion successfully resolves the anisotropy problem and allows for a simple evaluation (i.e., a solution of the first-order ordinary differential equation for a) of the electron-energy distribution function in highly anisotropic energy regions, characteristic of gases with relatively large inelastic cross sections.

APPENDIX: REDUCTION OF $R(\alpha,\varepsilon)$

We start from the definition (36) for $R(\alpha,\varepsilon)$ and, by changing the variables $\varepsilon-s=t$, transform it into

$$R(\alpha,\varepsilon)=e^{-\alpha\varepsilon}\int_0^\infty e^{\alpha t}\rho_t(\varepsilon-t)dt\ , \tag{A1}$$

where $\rho_t(\varepsilon-t)$ is given by [see Eq. (27)]

$$\rho_t(\varepsilon-t)=\frac{1}{2\sqrt{Bt}}\int_0^\infty du\frac{1}{u}e^{-u}\Psi\left[\frac{\varepsilon-t-Bu^2}{2\sqrt{Bt}\,u}\right]\ . \tag{A2}$$

We substitute (A2) into (A1) and use the definition of Ψ [Eq. (24)],

$$R(\alpha,\varepsilon)=\frac{1}{4\sqrt{B}}e^{-\alpha\varepsilon}\int_0^\infty du\frac{1}{u}e^{-u}\int_{t_1}^{t_2}dt\frac{1}{\sqrt{t}}e^{\alpha t}\ , \tag{A3}$$

where

$$\sqrt{t_{1,2}}=\sqrt{\varepsilon}\mp\sqrt{B}\,u\ . \tag{A4}$$

Finally, by making substitution $v=\sqrt{t}$, we rewrite (A3) as

$$R(\alpha,\varepsilon)=\frac{1}{2\sqrt{B}}e^{-\alpha\varepsilon}\int_0^\infty du\frac{1}{u}e^{-u}\int_{\mu-d}^{\mu+d}e^{\alpha v^2}dv\ , \tag{A5}$$

where $\mu=\sqrt{\varepsilon}$ and $d=u\sqrt{B}$.

At this stage we seek an expansion of $R(\alpha,\varepsilon)$ in powers of $\sqrt{B}\sim\delta$. This goal is accomplished by observing that with $F(v)$ defined as $F(v)=\int f(v)dv$, we can write

$$\begin{aligned}\int_{\mu-d}^{\mu+d}f\,dv&=F(\mu+d)-F(\mu-d)\\&=2\sum_{n=0}^{\infty}\frac{d^{2n+1}}{(2n+1)!}F^{(2n+1)}\Big|_{v=\mu}\\&=2\sum_{n=0}^{\infty}\frac{d^{2n+1}}{(2n+1)!}f^{(2n)}\Big|_{v=\mu}\ .\end{aligned} \tag{A6}$$

In the case of interest [see Eq. (A5)] $f=\exp(\alpha v^2)$, so that

$$f^{(l)}=[(2\alpha v)^l+l(l-1)\alpha(2\alpha v)^{l-2}+O(\alpha^{l-2})]f\ ,\quad l\ge 0\ . \tag{A7}$$

Note that $\alpha\sim 1/\delta$ and therefore the last expression displays the first two leading terms in the expansion in powers of δ. We use now (A7) in (A6) and substitute the resulting expression into (A5),

$$\begin{aligned}R&=\sum_{n=0}^{\infty}\frac{B^n}{(2n+1)!}\int_0^\infty du\frac{1}{u}e^{-u}u^{2n+1}f^{(2n)}\\&=\sum_{n=0}^{\infty}\frac{B^n}{2n+1}[(2\alpha\sqrt{\varepsilon})^{2n}+2n(2n-1)\alpha(2\alpha\sqrt{\varepsilon})^{2n-2}]\\&\quad+O(\delta^2)\ .\end{aligned} \tag{A8}$$

Finally, by defining

$$\alpha'=2\sqrt{B\varepsilon}\alpha \tag{A9}$$

and

$$S=\sum\frac{(\alpha')^{2n}}{2n+1}=\frac{1}{2\alpha'}\ln\left|\frac{1+\alpha'}{1-\alpha'}\right|\ , \tag{A10}$$

we can rewrite (A8) in the following compact form:

$$R=S+\frac{\alpha}{4\varepsilon}\frac{\partial^2 s}{\partial\alpha^2}+O(\delta^2)\ . \tag{A11}$$

[1]I. P. Shkarofsky, T. W. Jonston, and M. P. Bachynski, *The Particle Kinetics in Plasma* (Addison-Wesley, Reading, Mass., 1966), p. 71.
[2]T. Holdstein, Phys. Rev. **70**, 367 (1946).
[3]W. L. Nighan, Phys. Rev. A **2**, 1989 (1970).
[4]L. Friedland, Ph.D. thesis, Hebrew University of Jerusalem, 1976.
[5]W. H. Long, Jr., Air Force Aero Propulsion Laboratory Report No. AFAPL-TR-79-2038 (unpublished).
[6]L. Friedland, Phys. Fluids **20**, 1461 (1977).
[7]L. C. Pitchford, S. V. ONeil, and J. R. Rumble, Jr., Phys. Rev. A **23**, 294 (1981).
[8]L.C. Pitchford and A. V. Phelps, Phys. Rev. A **25**, 540 (1982).
[9]I. D. Reid, Aust. J. Phys. **32**, 231 (1979).
[10]S. Yoshida, A. V. Phelps, and L. C. Pitchford, Phys. Rev. A **27**, 2858 (1983).
[11]A. V. Phelps and L. C. Pitchford, Phys. Rev. A **31**, 2932 (1985).
[12]N. Ikuta and Y. Murakami, J. Phys. Soc. Jpn. **56**, 115 (1987).
[13]S. L. Lin and I. N. Bardsley, Comput. Phys. Commun. **15**, 161 (1978).
[14]J. Spanier and E. M. Gelbard, *Monte Carlo Principles and Neutron Transport Problems* (Addison-Wesley, Reading, Mass., 1969), p. 124.