Gyroresonant absorption from congruent reduction of an anisotropic pressure fluid model

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A system of Maxwell, multifluid momentum and anisotropic pressure equations for a weakly inhomogeneous magnetized plasma is written in a special Hermitian form. A recently developed congruent reduction technique, for extracting embedded, multidimensional, pairwise mode conversion events, is applied in studying the electron gyroresonant absorption problem. The mode conversion from an electromagnetic mode to a fluid pressure mode near the fundamental and second harmonic gyrofrequency is interpreted as gyroresonant absorption. The transmission coefficient is found in an arbitrary three-dimensional plasma and magnetic field geometry, demonstrating the potential of the approach for systematically reducing the order in general multicomponent wave propagation problems in nonuniform plasmas.

I. INTRODUCTION

Gyroresonant absorption problems at both the electron and ion cyclotron frequencies and their harmonics in a nonuniform plasma belong to a class of problems typically requiring lengthy and complex calculations.[1] The complexity is due to the fact that the absorption process basically is a thermal effect and therefore the theory necessarily involves solutions of the system of Maxwell and kinetic equations, a difficult task in usually three-dimensional plasma geometries. Even when the spatial variation of the equilibrium plasma parameters is sufficiently weak and the description of the waves via the geometric optics seems to be feasible, one cannot use the conventional geometric optics theory directly, since typically the gyroresonant absorption process is highly localized in space. This fact, in some sense, simplifies the theory, since various slab models can be used in studying the details of the interaction. Nevertheless, even the one-dimensional calculations performed to date have been rather complex and certainly very different in each particular application.

On the other hand, the recent developments of the congruent reduction[2] and multidimensional mode conversion[3] theories provide new tools in simplifying multicomponent wave propagation problems in plasmas. These theories comprise a systematic method of solving homogeneous integral equations of the form

$$\int d^4x'\, \hat{D}_{ij}(x,x')Z_j(x') = 0 \tag{1}$$

for an N-component vector field $\mathbf{Z}(x)$ on space-time $[x=(\mathbf{r},t)]$, where the dispersion kernel $\hat{\mathbf{D}}$ is Hermitian $[\hat{D}_{ij}(x,x') = \hat{D}^*_{ji}(x',x)]$ and known. In cases when $\mathbf{Z}$ describes a perturbation of a smooth, but weakly varying background equilibrium, the dependence of $\hat{\mathbf{D}}$ on $(x+x')/2$ is weak compared to that on $x-x'$, and the eikonal representation $\mathbf{Z}(x) = \mathbf{A}(x)\exp[i\psi(x)]$ with slowly varying amplitude $\mathbf{A}$ and rapidly varying phase ψ is feasible. We can then rewrite (1) as the slow amplitude transport equation

$$\mathbf{D}\cdot\mathbf{A} = i\left[\frac{\partial \mathbf{D}}{\partial k_\mu}\cdot\frac{\partial \mathbf{A}}{\partial x^\mu} + \frac{1}{2}\frac{d}{dx^\mu}\left(\frac{\partial \mathbf{D}}{\partial k_\mu}\right)\cdot\mathbf{A}\right] + O(\delta^2), \tag{2}$$

where

$$k_\mu(x) = \frac{\partial\psi}{\partial x^\mu},$$

$$\mathbf{D}(k,x) = \int d^4s\, \hat{\mathbf{D}}\left(x-\frac{s}{2}, x+\frac{s}{2}\right)\exp(-ik\cdot s),$$

and $\delta \ll 1$ is a small dimensionless parameter associated with the nonuniformity of the background. Our understanding of the solutions of Eq. (2) has advanced significantly with the development of the above-mentioned reduction and mode conversion theories. The congruent reduction theory[2] yields an algorithm for eliminating some of the components of $\mathbf{A}$ from the problem by preserving, at the same time, the form, first differential order, and Hermiticity of the *reduced* transport equation, which describes the remaining wave components. This reduced equation still has form (2) with $\mathbf{D}$ replaced by reduced matrix $\mathbf{D}^r$ of rank $M<N$, such that all its elements are of $O(\delta)$. Typically, in nondegenerate plasma regions, where only one of the eigenvalues of $\mathbf{D}(k,x)$ vanishes at a time, $M=1$ and thus the system reduces to a single first-order partial differential equation (PDE) for the last remaining component of the amplitude, so that the problem becomes easily integrable. In a more restricted class of problems, the final reduced matrix has rank $M=2$ and describes the pairwise mode conversion problem, the solution of which for a general geometry was found in Ref. 3. The total wave-action flux is still conserved in this type of problem (degenerate plasma regions), but nevertheless the flux is redistributed in two distinct channels (modes) automatically prescribed by the reduction procedure. This redistribution phenomenon is typically localized, so that outside the regions of the near degeneracy of $\mathbf{D}^r$, the reduction to case

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$M = 1$ is again possible and the simple integrability is restored for each of the two channels independently.

Thus if a physical problem is described, in its unreduced form, by Eq. (2), the solution can be found systematically by combining the reduction and multidimensional mode conversion theories. We therefore conclude that the problem of finding the solution in practice can be replaced by the question of whether a particular application allows description in form (2). We shall use the term "Hermitian problem" if the answer to this question is positive and show that the multispecies, anisotropic pressure magnetized plasma fluid model, in its unreduced form, comprises a Hermitian problem (see Sec. III). Thus, within this model, *all* wave problems in weakly nonuniform plasmas can be dealt with *systematically*, as described above. As an example, the actual reduction for the second harmonic and fundamental electron gyroresonance cases will be carried out in Sec. V and the results interpreted as the mode conversion from the electromagnetic to plasma pressure-fluid modes.

II. HERMITIAN FORM FOR COLD MULTISPECIES PLASMA MODEL

It is instructive to find the Hermitian form (2) for *unreduced* wave problems in the *cold* plasma case first. We start with the linearized Maxwell-momentum equations

$$\nabla\times\mathbf{E}_1 = -\frac{1}{c}\frac{\partial \mathbf{B}_1}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{B}_1 = \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi e}{c}\sum_\alpha Z_\alpha \epsilon_\alpha N_{\alpha 0}\mathbf{V}_{\alpha 1}, \tag{4}$$

$$m_\alpha \frac{\partial \mathbf{V}_{\alpha 1}}{\partial t} = Z_\alpha \epsilon_\alpha e\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_{\alpha 1}\times\mathbf{B}_0\right), \tag{5}$$

where m_α, Z_α, and ϵ_α are the mass, the charge number, and its sign, respectively, for species α, and $\mathbf{B}_0$ and $N_{\alpha 0}$ are the equilibrium magnetic field and density. We multiply Eq. (5) by $(N_{\alpha 0})^{1/2}$ (note that $\partial N_{\alpha 0}/\partial t = 0$) and define vector

$$\mathbf{Z} \equiv \begin{bmatrix} c(4\pi)^{-1/2}\mathbf{B}_1 \\ c(4\pi)^{-1/2}\mathbf{E}_1 \\ (N_{\alpha 0} m_\alpha)^{1/2}\mathbf{V}_{\alpha 1}\end{bmatrix} \equiv \mathrm{Re}\left(\begin{bmatrix}\mathbf{b}\\ \mathbf{a}\\ \mathbf{v}_\alpha\end{bmatrix}\exp(i\psi)\right). \tag{6}$$

Now, by inspecting Eqs. (3)–(5), we find that indeed, without any further approximation, the amplitude $\mathbf{A} \equiv (\mathbf{b},\mathbf{a},\mathbf{v}_\alpha)$ is described by Eq. (2), where the *Hermitian* dispersion matrix is given by

$$\mathbf{D} = \begin{array}{cc} \begin{array}{ccccccccc} b_x & b_y & b_z & a_x & a_y & a_z & v_{\alpha x} & v_{\alpha y} & v_{\alpha z} \end{array} & \\ \begin{bmatrix} \omega & 0 & 0 & 0 & k_z & -k_y & 0 & 0 & 0 \\ 0 & \omega & 0 & -k_z & 0 & k_x & 0 & 0 & 0 \\ 0 & 0 & \omega & k_y & -k_x & 0 & 0 & 0 & 0 \\ 0 & -k_z & k_y & \omega & 0 & 0 & i\omega_{p\alpha} & 0 & 0 \\ k_z & 0 & -k_x & 0 & \omega & 0 & 0 & i\omega_{p\alpha} & 0 \\ -k_y & k_x & 0 & 0 & 0 & \omega & 0 & 0 & i\omega_{p\alpha} \\ 0 & 0 & 0 & -i\omega_{p\alpha} & 0 & 0 & \omega & -i\Omega_{\alpha z} & i\Omega_{\alpha y} \\ 0 & 0 & 0 & 0 & -i\omega_{p\alpha} & 0 & i\Omega_{\alpha z} & \omega & -i\Omega_{\alpha x} \\ 0 & 0 & 0 & 0 & 0 & -i\omega_{p\alpha} & -i\Omega_{\alpha y} & i\Omega_{\alpha x} & \omega \end{bmatrix} & \begin{array}{c} b_x \\ b_y \\ b_z \\ a_x \\ a_y \\ a_z \\ v_{\alpha x} \\ v_{\alpha y} \\ v_{\alpha z} \end{array} \end{array}. \tag{7}$$

Here a *fixed* Cartesian coordinate system is used and the corresponding amplitude components are shown above and beside the matrix for easier identification of various matrix components. Also, we use definitions $c\omega = -\partial\psi/\partial t$, $c\omega_p = \pm(4\pi e^2 Z_\alpha^2 N_{\alpha 0}/m_\alpha)^{1/2}$, and $c\mathbf{\Omega}_\alpha = \pm eZ_\alpha \mathbf{B}_0/m_\alpha c$, where the sign is defined by the charge sign of the corresponding species. Note that all the frequencies differ by a factor of $1/c$ from the conventional definitions, which is convenient since now ω and $\mathbf{k}$ have the same dimensions. Thus we have demonstrated that the unreduced cold plasma case comprises a Hermitian problem, and therefore, within the model, various wave problems can be systematically reduced as described above and therefore solved in principle. Examples of such a reduction can be found elsewhere.[2]

At this point we shall discuss applications with the background magnetic field $\mathbf{B}_0(x)$, which defines a local preferred direction in the plasma. The dispersion matrix $\mathbf{D}$ has the form given by Eq. (7) in any *fixed* Cartesian coordinate system. We can ask what the effect on the form of the transport equation is, if, instead of constant base vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, we choose a different orthogonal representation, say $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$) associated, for example, with the direction of the magnetic field and thus dependent on position. The answer to this question can be found in Ref. 2. Indeed, the use of the new representation is equivalent to introducing a linear transformation of the amplitude $\mathbf{A}$, i.e.,

$$\mathbf{A}(x) = \mathbf{Q}(x)\cdot\overline{\mathbf{A}}(x), \tag{8}$$

where $\mathbf{Q}$ is the transformation matrix. It was shown in Ref. 2 that then to $O(\delta)$ the transformed amplitude $\overline{\mathbf{A}}$ is described by the transport equation of the same form as (2) with $\mathbf{D}$ replaced by $\overline{\mathbf{D}}$ given by

$$\overline{D}_{ij}(z) = Q^*_{mi}D_{mn}Q_{nj} + (i/2)(\{Q^*_{mi},D_{mn}\}Q_{nj} + Q^*_{mi}\{D_{mn},Q_{nj}\} + \{Q^*_{mi},Q_{nj}\}D_{mn}), \tag{9}$$

where $z = (k,x)$ and $\{F,G\} = (\partial F/\partial x^\mu)(\partial G/\partial k_\mu)$

$-(\partial G/\partial x^{\mu})(\partial F/\partial k_{\mu})$ is the conventional Poisson bracket. Now we can finally answer the question about the effect of introducing a preferred coordinate system. Since the nontrivial part of $\overline{\mathbf{D}}$ with the Poisson brackets is Hermitian and of $O(\delta)$, we can neglect its effect on the transport of the amplitude and assume that $\overline{\mathbf{D}}$ is simply the congruent transformation of $\mathbf{D}$:

$$\overline{\mathbf{D}} = \mathbf{Q}^{+}\cdot\mathbf{D}\cdot\mathbf{Q}. \tag{10}$$

Therefore, in conclusion, Eq. (2) with $\mathbf{D}$ given by (7) holds in *any slowly varying* Cartesian coordinate system. The convenient choice is the system with $\mathbf{e}_z$ along the direction of the background magnetic field $\mathbf{B}_0$. One can, of course, use a non-Cartesian coordinate system, such as the conventional base vectors $\mathbf{e}_{\pm} = (\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$ and $\mathbf{e}_z = \mathbf{B}_0/|\mathbf{B}_0|$. In this case we write $\mathbf{S} = S_+\mathbf{e}_+ + S_-\mathbf{e}_- + S_z\mathbf{e}_z$, where $\mathbf{S}$ is either $\mathbf{b}$, $\mathbf{a}$, $\mathbf{v}_\alpha$, or $\mathbf{k}$. Then, in the new representation, the evolution of the amplitude $\mathbf{A} = (b_+, b_-, b_z, a_+, a_-, a_z, v_{\alpha+}, v_{\alpha-}, v_{\alpha z})$ is described again by Eq. (2) with the dispersion matrix given by

$$\mathbf{D} = \begin{array}{c} \begin{array}{ccccccccc} b_+ & b_- & b_z & a_+ & a_- & a_z & v_{\alpha+} & v_{\alpha-} & v_{\alpha z} \end{array} \\ \begin{bmatrix} \omega & 0 & 0 & -ik_z & 0 & ik_+ & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & ik_z & -ik_- & 0 & 0 & 0 \\ 0 & 0 & \omega & ik_- & -ik_+ & 0 & 0 & 0 & 0 \\ ik_z & 0 & -ik_+ & \omega & 0 & 0 & i\omega_{p\alpha} & 0 & 0 \\ 0 & -ik_z & ik_- & 0 & \omega & 0 & 0 & i\omega_{p\alpha} & 0 \\ -ik_- & ik_+ & 0 & 0 & 0 & \omega & 0 & 0 & i\omega_{p\alpha} \\ 0 & 0 & 0 & -i\omega_{p\alpha} & 0 & 0 & \omega+\Omega_\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\omega_{p\alpha} & 0 & 0 & \omega-\Omega_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\omega_{p\alpha} & 0 & 0 & \omega \end{bmatrix} \end{array} \begin{array}{c} \\ b_+ \\ b_- \\ b_z \\ a_+ \\ a_- \\ a_z \\ v_{\alpha x} \\ v_{\alpha y} \\ v_{\alpha z} \end{array}, \tag{11}$$

where $\Omega_\alpha = \pm|\Omega_\alpha|$ with the signs again defined by the charge sign of species α.

Finally, we observe that as the result of the Hermiticity of $\mathbf{D}$, Eq. (2) yields the conservation law, i.e.,

$$\frac{\partial}{\partial x^{\mu}} j^{\mu} = 0, \quad j^{\mu} = \mathbf{A}^{*}\cdot\frac{\partial \mathbf{D}}{\partial k_{\mu}}\cdot\mathbf{A} + O(\delta). \tag{12}$$

In the cold plasma case described by dispersion matrix (7), Eq. (12) becomes

$$\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{G} = 0, \tag{13}$$

where

$$U = (\langle B_1^2\rangle + \langle E_1^2\rangle)/8\pi + N_{\alpha 0} m_\alpha \langle V_1^2\rangle/2,$$
$$\mathbf{G} = c\langle \mathbf{E}_1 \times \mathbf{B}_1\rangle/16\pi,$$

and $\langle\cdots\rangle$ describes the averaging over the rapid phase ψ. Thus we simply obtained the energy conservation law for the perturbed fluid.

III. THE ANISOTROPIC PRESSURE MODEL

Addition of thermal effects modifies the momentum equations for various species, which now become

$$m_\alpha\left(\frac{\partial}{\partial t} + \mathbf{V}_\alpha\cdot\nabla\right)\mathbf{V}_\alpha = \pm eZ_\alpha\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_\alpha\times\mathbf{B}\right) - \frac{1}{N_\alpha}\nabla\cdot\mathbf{P}_\alpha, \tag{14}$$

where the anisotropic pressure tensor $\mathbf{P}_\alpha$ evolves by[4]

$$\frac{\partial \mathbf{P}_\alpha}{\partial t} + \mathbf{P}_\alpha\cdot(\nabla\mathbf{V}_\alpha) + (\mathbf{P}_\alpha\cdot\nabla\mathbf{V}_\alpha)^T + \boldsymbol{\Omega}_\alpha\times\mathbf{P}_\alpha - \mathbf{P}_\alpha\times\boldsymbol{\Omega}_\alpha = 0. \tag{15}$$

We neglected the heat flux in the last equation, i.e., assumed a sufficiently low temperature case. The effect of collisions was also neglected for simplicity.

At this point we shall choose the simplest equilibrium, i.e., assume the Boltzmann distribution of the unperturbed density

$$N_{\alpha 0}(\mathbf{r}) = n_{\alpha 0}\exp[\pm eZ_\alpha\Phi(\mathbf{r})/T_\alpha], \tag{16}$$

with temperature T_α constant throughout the volume. Also, we shall use an isotropic pressure model in the equilibrium

$$\mathbf{P}_{\alpha 0} = N_{\alpha 0}(\mathbf{r})T_\alpha\mathbf{I}, \tag{17}$$

and assume no streaming in the fluid

$$\mathbf{V}_{\alpha 0} = 0. \tag{18}$$

It can be easily verified that Eqs. (16)–(18) indeed describe a good equilibrium of Eqs. (14) and (15).

Now we shall linearize the momentum and pressure equations

$$m_\alpha\frac{\partial \mathbf{V}_{\alpha 1}}{\partial t} = \pm Z_\alpha e\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_{\alpha 1}\times\mathbf{B}_0\right) - \frac{T_\alpha^{1/2}}{N_{\alpha 0}}\nabla\cdot\mathbf{P}'_{\alpha 1}, \tag{19}$$

$$\frac{\partial \mathbf{P}'_{\alpha 1}}{\partial t} + N_{\alpha 0}\,T_\alpha^{1/2}\mathbf{I}\nabla\cdot\mathbf{V}_{\alpha 1} + N_{\alpha 0}\,T_\alpha^{1/2}[(\nabla\mathbf{V}_{\alpha 1}) + (\nabla\mathbf{V}_{\alpha 1})^T] + \boldsymbol{\Omega}_{\alpha 0}\times\mathbf{P}'_{\alpha 1} - \mathbf{P}'_{\alpha 1}\times\boldsymbol{\Omega}_{\alpha 0} = 0. \tag{20}$$

Here we defined $\mathbf{P}'_{\alpha 1} = \mathbf{P}_{\alpha 1}/(T_\alpha)^{1/2}$. It can be seen from Eq. (20) that $\mathbf{P}'_{\alpha 1}$ is of $O(T_\alpha^{1/2})$. Let us make an additional ordering assumption, i.e., we shall treat the objects $kv_{\mathrm{th}\alpha}/\omega c \equiv (k/\omega c)(T_\alpha/m_\alpha)^{1/2}$, as being of $O(\delta)$. Then, to the lowest order, we can replace the operator ∇ in (19) and (20) by $i\mathbf{k}$ and $\partial/\partial t$ in (20) by $-i\omega$. The result is

$$m_\alpha \frac{\partial \mathbf{V}_{\alpha 1}}{\partial t} = \pm Z_\alpha e\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{V}_{\alpha 1}\times\mathbf{B}_0\right) - i\frac{T_\alpha^{1/2}}{N_{\alpha 0}}\mathbf{k}\cdot\mathbf{P}'_{\alpha 1}, \tag{21}$$

$$\omega\mathbf{P}'_{\alpha 1} + i(\boldsymbol{\Omega}_{\alpha 0}\times\mathbf{P}'_{\alpha 1} - \mathbf{P}'_{\alpha 1}\times\boldsymbol{\Omega}_{\alpha 0}) = N_{\alpha 0}\, T_\alpha^{1/2}(\mathbf{k}\cdot\mathbf{V}_{\alpha 1}\mathbf{I} + \mathbf{k}\mathbf{V}_{\alpha 1} + \mathbf{V}_{\alpha 1}\mathbf{k}). \tag{22}$$

In dealing with Eq. (22), at this stage, we replace the vector product in the left-hand side by the equivalent matrix operations, i.e., write

$$i(\boldsymbol{\Omega}_{\alpha 0}\times\mathbf{P}'_{\alpha 1} - \mathbf{P}'_{\alpha 1}\times\boldsymbol{\Omega}_{\alpha 0}) = \boldsymbol{\Omega}_\alpha\cdot\mathbf{P}'_{\alpha 1} + \mathbf{P}'_{\alpha 1}\cdot\boldsymbol{\Omega}_\alpha^T, \tag{23}$$

where the matrix $\boldsymbol{\Omega}_\alpha$ is defined via

$$\boldsymbol{\Omega}_\alpha = \begin{bmatrix} 0 & -i\Omega_{\alpha 0z} & i\Omega_{\alpha 0y} \\ i\Omega_{\alpha 0z} & 0 & -i\Omega_{\alpha 0x} \\ -i\Omega_{\alpha 0y} & i\Omega_{\alpha 0x} & 0 \end{bmatrix}. \tag{24}$$

It is convenient, at this stage, to introduce the representation in which $\boldsymbol{\Omega}_\alpha$ is diagonal. The desired base vectors are $\mathbf{e}_\pm$ and $\mathbf{e}_z$ as defined above. In terms of these vectors

$$\begin{aligned}\boldsymbol{\Omega}_\alpha &= \Omega_\alpha(\mathbf{e}_+\mathbf{e}_+^* - \mathbf{e}_-\mathbf{e}_-^*) \\ &= \Omega_\alpha(\mathbf{e}_+\mathbf{e}_- - \mathbf{e}_-\mathbf{e}_+),\end{aligned} \tag{25}$$

where $\Omega_\alpha = \pm|\boldsymbol{\Omega}_{\alpha 0}|$. In this representation, Eq. (22) is particularly simple and becomes written in components (we shall omit the subscript α for simplicity),

$$\begin{aligned}
(\omega-\Omega)P'_{z+} &= N_0T^{1/2}(k_zv_- + k_-v_z),\\
(\omega+\Omega)P'_{z-} &= N_0T^{1/2}(k_zv_+ + k_+v_-),\\
(\omega-2\Omega)P'_{-+} &= 2N_0T^{1/2}k_-v_-,\\
(\omega+2\Omega)P'_{+-} &= 2N_0T^{1/2}k_+v_+,\\
\omega P'_{xx} &= N_0T^{1/2}[2(k_+v_- + k_-v_+) + k_zv_z],\\
\omega P'_{zz} &= N_0T^{1/2}(k_+v_- + k_-v_+ + 3k_zv_z),
\end{aligned} \tag{26}$$

and for the remaining tensor elements

$$P_{-z} = P_{z+}, \quad P_{+z} = P_{z-}, \quad P_{--} = P_{++}. \tag{27}$$

Finally, we introduce new variables $\widetilde{P}_{++} = P_{++}/\sqrt{2}$ and $\widetilde{P}_{zz} = (2P_{zz} - P_{++})/\sqrt{10}$, and, similar to Eq. (6), use the eikonal representation, i.e., define the wave field $\mathbf{Z}$ via

$$\mathbf{Z} \equiv \begin{bmatrix} c\mathbf{B}_1/(4\pi)^{1/2} \\ c\mathbf{E}_1/(4\pi)^{1/2} \\ (N_0m)^{1/2}\mathbf{V}_1 \\ (TN_0)^{-1/2}P_{z+} \\ (TN_0)^{-1/2}P_{z-} \\ (TN_0)^{-1/2}P_{-+} \\ (TN_0)^{-1/2}P_{+-} \\ (TN_0)^{-1/2}\widetilde{P}_{++} \\ (TN_0)^{-1/2}\widetilde{P}_{zz} \end{bmatrix} \equiv \mathrm{Re}\left[\begin{pmatrix} \mathbf{b} \\ \mathbf{a} \\ \mathbf{v} \\ p_{z+} \\ p_{z-} \\ p_{-+} \\ p_{+-} \\ p_{++} \\ p_{zz} \end{pmatrix}\exp(i\psi)\right] \equiv \mathrm{Re}[\mathbf{A}\exp(i\psi)]. \tag{28}$$

Equations (3), (4), (21), and (26) can now be written in form (2) for the amplitude $\mathbf{A} = (\mathbf{b},\mathbf{a},\mathbf{v}_\alpha, p_{z+}, p_{z-}, p_{-+}, p_{+-}, p_{++}, p_{zz})$ with the Hermitian dispersion matrix given by [compare with Eq. (11) for the cold plasma case]

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{ba} & 0 & 0 \\ \mathbf{D}_{ba}^* & \mathbf{D}_{aa} & \mathbf{D}_{av} & 0 \\ 0 & \mathbf{D}_{av}^* & \mathbf{D}_{vv} & \mathbf{D}_{vp} \\ 0 & 0 & \mathbf{D}_{vp}^* & \mathbf{D}_{pp} \end{bmatrix}, \tag{29}$$

where the various matrices are

$$\mathbf{D}_{bb} = \mathbf{D}_{aa} = \omega\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_{vv} = \begin{bmatrix} \omega+\Omega & 0 & 0 \\ 0 & \omega-\Omega & 0 \\ 0 & 0 & \omega \end{bmatrix}; \tag{30}$$

$$\mathbf{D}_{av} = i\omega_p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_{ba} = \begin{bmatrix} -ik_z & 0 & ik_+ \\ 0 & ik_z & -ik_- \\ ik_- & -ik_+ & 0 \end{bmatrix}; \tag{31}$$

$$\mathbf{D}_{vp} = \begin{bmatrix} 0 & \beta k_z & 0 & \beta k_- & \sqrt{2}\beta k_+ & 0 \\ \beta k_z & 0 & \beta k_+ & 0 & \sqrt{2}\beta k_- & 0 \\ \beta k_+ & \beta k_- & 0 & 0 & \beta k_z/\sqrt{2} & \sqrt{5/2}\beta k_z \end{bmatrix}; \tag{32}$$

$$\mathbf{D}_{pp} = \begin{bmatrix} \omega-\Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega+\Omega & 0 & 0 & 0 & 0 \\ 0 & 0 & (\omega-2\Omega)/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\omega+2\Omega/2) & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix}. \tag{33}$$

Here $\beta = v_{\rm th}/c$ and we observe that the thermal effects are introduced via the coupling matrix $\mathbf{D}_{vp}$ between the fluid velocity and pressure tensor perturbations. This completes the derivation of the Hermitian form (2) for the anisotropic pressure fluid model.

We shall conclude this section with the following two remarks. The first is regarding the validity of the model. We expect the fluid description to be valid in cases when the phenomenon of interest involves interactions with the *bulk* of particle distributions. Therefore such effects as the Landau damping by the distribution tails are clearly omitted from the fluid description. On the other hand, in the vicinity of the cyclotron resonances where $\omega - n\Omega_\alpha \sim O(\delta)$, the bulk of the distribution of the corresponding species is in the resonance, since, by our ordering assumption, $kv_{\rm th}/\omega c \sim O(\delta)$. Therefore the fluid description of the resonant cyclotron interaction can be expected to be valid. This indeed will be demonstrated in the examples in Sec. VI by a direct comparison with the results of the kinetic theory.

The second remark is related to the energy conservation. Because of the Hermiticity of $\mathbf{D}$, as in the cold plasma case, the anisotropic pressure model yields the conservation law of form (13). This is, of course, an expected feature of the theory, since the collisional dissipation was neglected. The energy conservation is especially important in the context of possible localized linear mode conversion events in our multidegree of freedom system. The energy is conserved globally, still allowing the energy flux *redistribution* in various channels (modes) when the local linear mode conversion takes place.

IV. CONGRUENT REDUCTION

Before proceeding to various examples, let us briefly describe the details of the reduction procedure.[2] The method is applied as follows. Consider an unreduced, Hermitian, $N \times N$ dispersion matrix $\mathbf{D}$ characterizing a Hermitian problem, i.e., the case described by Eq. (2). If any *diagonal* element of $\mathbf{D}$, say D_{kk} is of $O(1)$, then the kth wave component A_k can be eliminated from the problem and the remaining $N-1$ components are again described by Eq. (2), with $\mathbf{D}$ replaced by the reduced $(N-1)\times(N-1)$ dispersion matrix $\mathbf{D}^r$ given by

$$D^r_{ij} = D_{ij} - D_{ik}D_{kj}/D_{kk}, \quad i,j \neq k. \tag{34}$$

This is the essence of the reduction theorem proved in Ref. 2. The case, when all the diagonal elements of $\mathbf{D}$ are of $O(\delta)$, but a nondiagonal element is of $O(1)$, still allows the reduction of order (even from N to $N-2$, in this case[2]). This situation, however, is very rare and will not be discussed here.

Equation (34) describes a single reduction step. The procedure is then repeated for the reduced problem described by $\mathbf{D}^r$ and so on, until one arrives at the final reduced dispersion matrix $\mathbf{D}^f$ with all elements of $O(\delta)$. The final transport equation still has form (2) but is irreducible within the geometric optics approximation. In the most simple doubly degenerate case, the rank of $\mathbf{D}^f$ is 2 and the transport equation describes the pairwise mode conversion problem.[3] Examples of this type of problem will be given in the next section.

We notice that according to Eq. (34), D^r_{ij} $(i,j\neq k)$ differs from the original dispersion matrix element only if *both* coupling elements D_{ik} and D_{kj} do not vanish. This is an important observation because of the sparsity of many unreduced dispersion matrixes [see, for example, Eq. (29)]. Indeed, we can shorten the reduction procedure, at each reduction step, by eliminating the amplitude component (say A_k) characterized by the smallest number of the nonvanishing coupling elements D_{ik} $(i\neq k)$. Three cases of increasing complexity are then encountered frequently.

Case 1: $D_{ik} = 0$ $(i\neq k)$. None of the components of the dispersion tensor are modified as the result of the reduction in this case, i.e., $D^r_{ij} = D_{ij}$ $(i,j\neq k)$, and component A_k simply *drops* from the problem $[A_k \sim O(\delta)]$.

Case 2: $D_{kl}\neq 0$, $D_{ki}=0$, and $i\neq l$ $(i,l\neq k)$. Only *one* matrix element changes in this case as the result of the elimination of A_k, i.e., the *diagonal* element $D^r_{ll} = D_{ll} - |D_{kl}|^2/D_{kk}$. All the remaining elements $D^r_{ij} = D_{ij}$ $(i,j\neq l)$.

Case 3: $D_{kl}\neq 0$, $D_{km}\neq 0$, $D_{ki}=0$, and $i\neq l,m$ $(i,l,m\neq k)$. In this case *four* elements of $\mathbf{D}$ are affected by the reduction, i.e., D_{ll}, D_{mm}, D_{lm}, and D_{ml}.

Thus, in conclusion, the reduction algorithm is relatively simple for sparse matrices and the algebraic complexity increases rapidly when the matrix becomes increasingly nonsparse as the result of the reduction. At this point we proceed to examples in which cases 1, 2, and 3 above are most frequently encountered in the step by step reduction process.

V. REDUCTION OF THE ANISOTROPIC PRESSURE MODEL AND GYRORESONANT ABSORPTION

We return now to the anisotropic pressure fluid case and consider the electron cyclotron frequency range, neglecting the ion contribution. Also, Eqs. (26) show that thermal effects are most important in the vicinity of the fundamental or second harmonic gyroresonances [i.e., where $\omega+\Omega$ or $\omega+2\Omega$ are of $O(\delta)$], since one expects components p_{z-} or

p_{+-} to be relatively large in these regions. Consequently, we shall neglect all the components of $\mathbf{p}$ but p_{z-} (or p_{+-} for the second harmonic case) in the wave amplitude [Eq. (28)]. We shall consider the second harmonic resonance first, so that the amplitude vector is $\mathbf{A} = (\mathbf{b},\mathbf{a},\mathbf{v},p_{+-})$, where $\mathbf{v}$ represents the perturbed electron fluid velocity. Additional simplification is achieved by restricting the treatment of the second harmonic to the perpendicular incidence case, i.e., $k_x = k$, and $k_y = k_z = 0$, described by the unreduced dispersion matrix [see Eq. (29)]:

$$\mathbf{D}=\begin{array}{cc} \begin{array}{cccccccccc} b_+ & b_- & b_z & a_+ & a_- & a_z & v_+ & v_- & v_z & p_{+-} \end{array} & \\ \left[\begin{array}{cccccccccc} \omega & 0 & 0 & 0 & 0 & ik/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & -ik/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & ik/\sqrt{2} & -ik/\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ik/\sqrt{2} & \omega & 0 & 0 & i\omega_p & 0 & 0 & 0 \\ 0 & 0 & ik/\sqrt{2} & 0 & \omega & 0 & 0 & i\omega_p & 0 & 0 \\ -ik/\sqrt{2} & ik/\sqrt{2} & 0 & 0 & 0 & \omega & 0 & 0 & i\omega_p & 0 \\ 0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega+\Omega & 0 & 0 & \beta k/\sqrt{2} \\ 0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega-\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta k/\sqrt{2} & 0 & 0 & (\omega+2\Omega)/2 \end{array}\right] & \begin{array}{c} b_+ \\ b_- \\ b_z \\ a_+ \\ a_- \\ a_z \\ v_+ \\ v_- \\ v_z \\ p_{+-} \end{array} \end{array}. \tag{35}$$

At this point we start the reduction procedure by focusing our attention on the vicinity of the second harmonic resonance, $\omega + 2\Omega \sim O(\delta)$. We eliminate the components b_+, b_-, v_-, and v_z first. These reduction steps correspond to case 2 in Sec. IV (only one nonvanishing coupling element in the dispersion matrix). The reduction in this case affects only the corresponding diagonal elements of the dispersion matrix. The reduced amplitude becomes $\mathbf{A} = (b_z,a_+,a_-,a_z,v_+,p_{+-})$ and the corresponding reduced dispersion matrix at this stage is

$$\mathbf{D}=\begin{array}{cc} \begin{array}{cccccc} b_z & a_+ & a_- & a_z & v_+ & p_{+-} \end{array} & \\ \left[\begin{array}{cccccc} \omega & \dfrac{ik}{\sqrt{2}} & \dfrac{-ik}{\sqrt{2}} & 0 & 0 & 0 \\ \dfrac{-ik}{\sqrt{2}} & \omega & 0 & 0 & i\omega_p & 0 \\ \dfrac{ik}{\sqrt{2}} & 0 & \omega-\dfrac{\omega_p^2}{\omega-\Omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega-\dfrac{k^2+\omega_p^2}{\omega} & 0 & 0 \\ 0 & -i\omega_p & 0 & 0 & \omega+\Omega & \dfrac{\beta k}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \dfrac{\beta k}{\sqrt{2}} & \dfrac{\omega+2\Omega}{2} \end{array}\right] & \begin{array}{c} b_z \\ a_+ \\ a_- \\ a_z \\ v_+ \\ p_{+-} \end{array} \end{array}. \tag{36}$$

Now, obviously, component a_z can be simply omitted (case 1 in Sec. IV) and a_- can be reduced as in case 2. The result is

$$D=\begin{array}{cc} \begin{array}{cccc} b_z & a_+ & v_+ & p_{+-} \end{array} & \\ \left[\begin{array}{cccc} \dfrac{(\omega-\Omega(2\omega^2-k^2)-2\omega\omega_p^2}{2(\omega^2-\omega\Omega-\omega_p^2)} & \dfrac{ik}{\sqrt{2}} & 0 & 0 \\ \dfrac{-ik}{\sqrt{2}} & \omega & i\omega_p & 0 \\ 0 & -i\omega_p & \omega+\Omega & \dfrac{\beta k}{\sqrt{2}} \\ 0 & 0 & \dfrac{\beta k}{\sqrt{2}} & \dfrac{\omega+2\Omega}{2} \end{array}\right] & \begin{array}{c} b_z \\ a_+ \\ v_+ \\ p_{+-} \end{array} \end{array}. \tag{37}$$

Next, we reduce b_z (case 2), which results in

$$\mathbf{D}=\begin{pmatrix}\frac{(\omega^2-k^2)[2\omega(\omega-\Omega)-\omega_p^2]-\omega_p^2\omega^2}{(\omega-\Omega)(2\omega^2-k^2)-2\omega\omega_p^2} & i\omega_p & 0\\ -i\omega_p & \omega+\Omega & \frac{\beta k}{\sqrt{2}}\\ 0 & \frac{\beta k}{\sqrt{2}} & \frac{\omega+2\Omega}{2}\end{pmatrix}, \tag{38}$$

describing $\mathbf{A}=(a_+,v_+,p_{+-})$. Finally elimination of a_+ (again case 2) yields the final characteristic matrix

$$\mathbf{D}^f=\begin{bmatrix}\frac{2\omega[(\omega^2-k^2)(\omega^2-\Omega^2-\omega_p^2)-\omega_p^2(\omega^2-\omega_p^2)]}{(\omega^2-k^2)[2\omega(\omega-\Omega)-\omega_p^2]-\omega_p^2\omega^2} & \frac{\beta k}{\sqrt{2}}\\ \frac{\beta k}{\sqrt{2}} & \frac{\omega+2\Omega}{2}\end{bmatrix}\equiv\begin{bmatrix}D_a & \eta\\ \eta^* & D_b\end{bmatrix} \tag{39}$$

for the remaining amplitude components $\mathbf{A}=(v_+,p_{+-})$. We observe that the complete near degeneracy of this matrix requires

$$D_a=\frac{2\omega[(\omega^2-k^2)(\omega^2-\Omega^2-\omega_p^2)-\omega_p^2(\omega^2-\omega_p^2)]}{(\omega^2-k^2)[2\omega(\omega-\Omega)-\omega_p^2]-\omega_p^2\omega^2}\sim O(\delta), \tag{40}$$

$$D_b=(\omega+2\Omega)/2\sim O(\delta). \tag{41}$$

If (40) and (41) are satisfied, a further reduction of $\mathbf{D}^f$ yields singular coefficients and therefore is forbidden within the geometric optics approximation. In this irreducible, near-degenerate case, the corresponding final transport equation (2) with $\mathbf{D}$ replaced by $\mathbf{D}^f$ serves as the system of coupled mode equations,[3] describing two weakly coupled modes, given by $D_a\approx 0$ (this is the cold plasma extraordinary mode dispersion relation at the resonance) and $D_b=(\omega+2\Omega)/2\approx 0$, which is the fluid pressure mode carried by the component p_{+-} of the pressure tensor perturbation. The coupling is due to a small thermal effect and $\eta=\beta k/\sqrt{2}$ serves as a weak coupling coefficient, almost constant throughout the region (normal degeneracy[2]).

Now we can find the transmission coefficient of the extraordinary mode through the mode coupling region. The transmission, for general geometry, is given by[3]

$$T=\exp(-2\pi|\eta|^2/|B|), \tag{42}$$

where

$$B=\{D_a,D_b\}_{x_0,k_0}=\left(\frac{\partial D_a}{\partial x^\mu}\frac{\partial D_b}{\partial k_\mu}-\frac{\partial D_b}{\partial x^\mu}\frac{\partial D_a}{\partial k_\mu}\right)_{x_0,k_0}, \tag{43}$$

and is evaluated at the *crossing point* $\mathbf{x}_0,\mathbf{k}_0$ ($D_a=D_b=0$ at $\mathbf{x}_0,\mathbf{k}_0$), which is defined along the geometric optics ray generated by the dispersion $D_a=0$ (in our case the ray for the extraordinary cold plasma mode). Evaluation of T in the case of interest yields

$$T=\exp\left(-\frac{\pi\beta^2k^2\omega_p^2}{4|\mathbf{k}\cdot\nabla\Omega|}\times\frac{(\omega^2-k^2)[2\omega(\omega-\Omega)-\omega_p^2]-\omega^2\omega_p^2}{\omega(\omega^2-\Omega^2-\omega_p^2)}\right). \tag{44}$$

Note that since Eq. (44) is evaluated at the crossing point, where $D_a=D_b=0$, we can substitute $\omega=-2\Omega$ and

$$\omega^2-k^2=\omega_p^2(\omega^2-\omega_p^2)/(\omega^2-\Omega^2-\omega_p^2). \tag{45}$$

Then, after some algebra, the transmission coefficient becomes

$$T=\exp\left[-\frac{\pi\beta^2k^2\omega_p^2}{|\mathbf{k}\cdot\nabla\Omega|\omega}\left(\frac{3\omega^2-2\omega_p^2}{3\omega^2-4\omega_p^2}\right)^2\right]. \tag{46}$$

This expression, for the one-dimensional case and $\nabla\Omega\perp\mathbf{B}_0$ (perpendicular stratification of the magnetic field), coincides with the result predicted by a more elaborate kinetic theory (see the article by Antonsen and Manheimer cited in Ref. 1). The same expression for T in the one-dimensional case was also obtained by Cairns and Lashmore-Davies[5] (CLD) by means of a different approach to the mode conversion problem. Their method exploited the local dispersion relation by transforming it into a special characteristic form and associating the coupled mode equations with this form. The weakness of this approach is the ambiguity in choosing the proper *differentiation operator* in constructing the coupled mode system, because the information on the gradients of slowly varying plasma parameters is missing in the local dispersion relation. The ambiguity grows with the dimensionality of the problem and, as the result, only one-dimensional examples have been studied within the CLD formalism. Here, in contrast, Eq. (46) was derived by using the multidimensional reduction and mode conversion theories,[2,3] and therefore is not limited to the one-dimensional case and can be used in plasmas of arbitrary geometry.

At this stage, after demonstrating the reduction method in a relatively simple case (the second harmonic at the perpendicular incidence), we proceed to the fundamental resonance in its *full complexity*, i.e., for a general magnetogeometry and an arbitrary direction of propagation. We now retain the component p_{z-} instead of p_{+-} used in the previous case, and therefore the unreduced amplitude becomes $\mathbf{A}=(\mathbf{b},\mathbf{a},\mathbf{v},p_{z-})$. The corresponding dispersion matrix is [see Eq. (29)]

$$
\mathbf{D}=\begin{array}{c}
\begin{array}{cccccccccc} b_+ & b_- & b_z & a_+ & a_- & a_z & v_+ & v_- & v_z & p_{z-} \end{array}\\
\left[\begin{array}{cccccccccc}
\omega & 0 & 0 & -ik_z & 0 & ik_+ & 0 & 0 & 0 & 0\\
0 & \omega & 0 & 0 & ik_z & -ik_- & 0 & 0 & 0 & 0\\
0 & 0 & \omega & ik_- & -ik_+ & 0 & 0 & 0 & 0 & 0\\
ik_z & 0 & -ik_+ & \omega & 0 & 0 & i\omega_p & 0 & 0 & 0\\
0 & -ik_z & ik_- & 0 & \omega & 0 & 0 & i\omega_p & 0 & 0\\
-ik_- & ik_+ & 0 & 0 & 0 & \omega & 0 & 0 & i\omega_p & 0\\
0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega+\Omega & 0 & 0 & \beta k_z\\
0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega-\Omega & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 & \omega & \beta k_-\\
0 & 0 & 0 & 0 & 0 & 0 & \beta k_z & 0 & \beta k_+ & \omega+\Omega
\end{array}\right]
\end{array}
\begin{array}{c} \\ b_+\\ b_-\\ b_z\\ a_+\\ a_-\\ a_z\\ v_+\\ v_-\\ v_z\\ p_{z-} \end{array}. \tag{47}
$$

The reduction of this matrix proceeds as follows. We eliminate v_- first (case 2 in Sec. IV) and, being interested in the resonant situation ($\omega+\Omega\approx 0$), set $\omega-\Omega=2\omega$ in this step. Next we reduce b_+, b_-, b_z, and v_z (all these reduction steps correspond to case 3 in Sec. IV). The reduced amplitude vector at this stage $\mathbf{A}=(a_+, a_-, a_z, v_+, p_{z-})$ is characterized by the reduced dispersion matrix

$$
\mathbf{D}=\begin{array}{c}
\begin{array}{ccccc} a_+ & a_- & a_z & v_+ & p_z \end{array}\\
\left[\begin{array}{ccccc}
\omega-(k_z^2+k_+k_-)/\omega & k_+^2/\omega & k_+k_z/\omega & i\omega_p & 0\\
k_-^2/\omega & \omega-(k_z^2+k_+k_-+\omega_p^2/2)/\omega & k_-k_z/\omega & 0 & 0\\
k_-k_z/\omega & k_+k_z/\omega & \omega-(2k_+k_-+\omega_p^2)/\omega & 0 & -i\omega_p\beta k_-/\omega\\
-i\omega_p & 0 & 0 & \omega+\Omega & \beta k_z\\
0 & 0 & i\omega_p\beta k_+/\omega & \beta k_z & \omega+\Omega-\beta^2k_+k_-/\omega
\end{array}\right]
\end{array}. \tag{48}
$$

Now we eliminate a_+, obtaining the matrix

$$
\mathbf{D}=\begin{array}{c}
\begin{array}{cccc} a_- & a_z & v_+ & p_{z-} \end{array}\\
\left[\begin{array}{cccc}
\omega-(k_z^2+k_+k_-)/\omega-k_+^2k_-^2/\omega^2A & k_-k_z/\omega-k_+k_-^2k_z/\omega A & -i\omega_pk_-^2/\omega A & 0\\
k_+k_z/\omega-k_-k_+^2k_z/\omega^2A & \omega-(2k_+k_-+\omega_p^2)/\omega-k_+k_-k_z^2/\omega^2A & -i\omega_pk_-k_z/\omega A & -i\omega_p\beta k_-/\omega\\
i\omega_pk_+^2/\omega A & i\omega_pk_+k_z/\omega A & \omega+\Omega-\omega_p^2/A & \beta k_z\\
0 & i\omega_p\beta k_+/\omega & \beta k_z & \omega+\Omega-\beta^2k_+k_-/\omega
\end{array}\right]
\end{array}, \tag{49}
$$

where $A=\omega-(k_z^2+k_+k_-)/\omega=\omega(2-2n_z^2-n_\perp^2)$ and $\mathbf{n}=\mathbf{k}/\omega$. The next step is to reduce v_+. In doing so we can set $\omega+\Omega-\omega_p^2/A\approx-\omega_p^2/A$. The resulting matrix is

$$
\mathbf{D}=\begin{array}{c}
\begin{array}{ccc} a_- & a_+ & p_{z-} \end{array}\\
\left[\begin{array}{ccc}
\omega-(k_z^2+k_+k_-+\omega_p^2/2)\omega & k_-k_z/\omega & -ik_-^2k_z\beta/\omega\omega_p\\
k_+k_z/\omega & \omega-(2k_+k_-+\omega_p^2)/\omega & -i\omega_pk_-\beta(1+k_z^2/\omega_p^2)/\omega\\
ik_+^2k_z\beta/\omega\omega_p & i\omega_pk_+\beta(1+k_z^2/\omega_p^2)/\omega & \omega+\Omega+O(\beta^2)
\end{array}\right]
\end{array}
\begin{array}{c} \\ a_-\\ a_z\\ p_{z-} \end{array}. \tag{50}
$$

Finally we eliminate a_- and obtain the standard final (2×2) matrix

$$\mathbf{D}^f=\begin{bmatrix} D_a & \eta\\ \eta^* & D_b \end{bmatrix}, \tag{51}$$

describing the final reduced amplitude $A^f=(a_z, p_{z-})$. The components of this matrix are

$$D_a=\omega(1-n_\perp^2-\alpha^2-n_\perp^2n_z^2/A'), \tag{52}$$

$$D_b=\omega+\Omega+O(\beta^2), \tag{53}$$

$$\eta=-i\beta\alpha k_-\,[\,1+(n_z^2/\alpha^2)\,(\,1-n_\perp^2/A')\,], \tag{54}$$

where $\alpha^2=\omega_p^2/\omega^2$ and $A'=2-2n^2+n_\perp^2-\alpha^2$.

The complete near degeneracy of matrix $\mathbf{D}^f$ requires that simultaneously

$$D_a=(\omega/A')\,[\,(1-\alpha^2)(2-2n^2-\alpha^2)-n_\perp^2(1-n^2)\,]\sim O(\delta), \tag{55}$$

$$D_b=\omega+\Omega+O(\beta^2)\sim O(\delta), \tag{56}$$

$$\eta\sim O(\delta). \tag{57}$$

The last two conditions are already satisfied, since $\omega+\Omega\sim O(\delta)$ and $\beta\sim O(\delta)$. We also observe that

$$(1-\alpha^2)(2-2n^2-\alpha^2)-n_\perp^2(1-n^2)=0 \tag{58}$$

is the general, high frequency, cold plasma dispersion relation at the cyclotron resonance.[6] Therefore, in the degenerate case, $\mathbf{D}^f$ describes the weak coupling between the cold plasma (ordinary or extraordinary) modes ($D_a \approx 0$) and the pressure mode ($D_b \approx 0$) associated with p_{z-} component of the perturbed pressure tensor. The off-diagonal element η serves as a small coupling coefficient in this linear mode conversion process. A simple expression for η can be found by observing that the cold plasma dispersion relation yields

$$1/A' = (1 - n_\perp^2 - \alpha^2)/n_\perp^2\, n_z^2, \tag{59}$$

so that (54) can be rewritten as

$$\eta = -(i\beta k_-/\alpha)(2\alpha^2 + n^2 - 1). \tag{60}$$

At this point we evaluate the transmission coefficient [see Eq. (42)]:

$$T = \exp\left(-\frac{2\pi|\eta|^2}{|B|}\right) = \exp\left(-\frac{\pi\beta^2 k_\perp^2 (2\alpha^2 + n^2 - 1)^2}{\alpha^2 |\partial D_a/\partial \mathbf{k}\cdot\nabla\Omega|}\right). \tag{61}$$

This expression, as usual, should be evaluated at the crossing point ($D_a = D_b = 0$) along the cold plasma ray, characterizing the incident wave. Note that Eq. (61) describes an arbitrary magnetic field stratification (the direction of $\nabla\Omega$ relative to $\mathbf{B}_0$), as well as a general direction of propagation of the incident cold plasma mode. To the best of our knowledge, this is the first general compact expression for the transmission in the problem of interest, obtained entirely within the mode conversion theory.

In order to check our result, we shall rewrite Eq. (61) in the form

$$T = \exp\left(-\frac{\pi\beta^2 k_\perp^2 (2\alpha^2 + n^2 - 1)^2}{\alpha^2 |\partial D_a/\partial \mathbf{k}_\perp||\nabla\Omega|}\frac{\sin\theta}{\cos\gamma}\right), \tag{62}$$

where θ and γ are the angles between the direction of the cold plasma ray (direction of $\partial D_a/\partial \mathbf{k}$) at the resonance and the directions of $\mathbf{B}_0$ and $\nabla\Omega$, respectively. The direct differentiation of Eq. (52) and use of (59) yield

$$\frac{\partial D_a}{\partial \mathbf{k}_\perp} = -\frac{2\mathbf{n}_\perp}{n_\perp^2 n_z^2}\left[(1 - n_\perp^2 - \alpha^2)^2 + n_z^2(1-\alpha^2)\right]. \tag{63}$$

Also, from (58),

$$2\alpha^2 + n^2 - 1 = -\left[(2 - 2n_z^2 - \alpha^2)(1-\alpha^2) - 2n_\perp^2\right]/n_\perp^2. \tag{64}$$

Then Eq. (62) can be written as

$$T = \exp\left(-\frac{\pi T\omega^2 n_z^2 \sin\theta}{2mc^2\alpha^2 n_\perp |\nabla\Omega|\cos\gamma} \times \frac{\left[(2 - 2n_z^2 - \alpha^2)(1-\alpha^2) - 2n_\perp^2\right]^2}{(1 - n_\perp^2 - \alpha^2)^2 + n_z^2(1-\alpha^2)}\right), \tag{65}$$

again in full agreement with the results based on the kinetic theory.[7]

VI. CONCLUSIONS

(i) It was shown that the unreduced anisotropic pressure, multifluid plasma model can be formulated in the Hermitian form (2), characterized by dispersion matrix (29). Consequently, within the model, all waves in a weakly varying plasma can be dealt with within the framework of the congruent reduction method.[2] This method *automatically* yields either a single PDE describing one of the wave components in the nondegenerate plasma case, or a system of coupled mode equations (two coupled PDE's) in nearly degenerate plasma regions, where the mode conversion takes place. Thus the degenerate case reduces to the multidimensional mode conversion problem, the solution of which is already known for general geometry.[3]

(ii) Examples of application of the reduction algorithm in the vicinity of the fundamental and second harmonic electron gyroresonances, in Sec. V, demonstrate our systematic approach to the reduction of order. The procedure yields the characteristic coupled mode system at the resonances suggesting the interpretation of the gyroresonant absorption as the mode conversion from the cold plasma modes to a fluid-pressure mode. The multidimensional mode conversion theory[3] was applied to this system and allowed to find compact expressions for the transmission of the cold plasma modes through the gyroresonances in a general three-dimensional geometry and (in the case of the fundamental resonance) for arbitrary direction of propagation of the incident wave.

(iii) We have considered a single species case in the examples of Sec. V. Nevertheless, since the method is general, it can be automated by using a computer, thus allowing the study of much more complicated mode interactions, such as those in multispecies plasmas of arbitrary geometry, where the usual theories are typically restricted to slab models and become extremely elaborate and nontrivial.

(iv) The possibility of formulating both the cold plasma and the anisotropic pressure fluid models in the Hermitian form suggests that higher moment equations, and possibly the kinetic problem in the absence of collisions, also comprise the Hermitian case. We shall address this problem in our future studies.

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