Spatial autoresonance: Enhancement of mode conversion due to nonlinear phase locking

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It is shown that the inclusion of weak nonlinearities may have a profound effect on the conservative linear mode conversion in inhomogeneous media. The phenomenon is demonstrated by adding a quadratic nonlinearity to the conventional system of integral equations describing spatial evolution of linear multicomponent waves. The nonlinear generalization of the reduced coupled mode equations then shows the possibility of a persistent interaction between the modes via the "spatial autoresonance," i.e., the spatial self-adjustment of the nonlinear resonance condition. For a sufficiently strong nonlinearity this adjustment either (a) discontinues at some point when both interacting modes have the same action density sign and the initially excited mode transfers its action flux to the second mode, or (b) prevails indefinitely, in the case of positive–negative action density mode couplings, as the magnitudes of both modes grow continuously in space. The conditions for the spatial autoresonance are found and the effect is illustrated by numerical examples.

I. INTRODUCTION

Linear mode conversion is a phenomenon characteristic of multicomponent waves propagating in a weakly nonuniform background. It has been studied extensively in the past in the context of waves in ionospheric plasmas[1] and now attracts a renewed attention as the main mechanism in various plasma heating schemes.[2] This recent interest has stimulated the development of general theoretical approaches for dealing with the linear mode conversion phenomenon in space and time varying plasmas of arbitrary magnetogeometries.[3,4] The new theories allow one to extract conservative pairwise linear mode couplings embedded in high-order multicomponent linear systems of integral equations of the form

$$\int d^4x' \, \mathscr{D}_{ij}(x,x') Z_j(x') = 0, \tag{1}$$

describing evolution of real N-component perturbations $Z_i(x)$ ($i=1,2,...,N$) defined on space-time $[x=(\mathbf{r},t)]$ in a weakly inhomogeneous and time-dependent background characterized by a symmetric dispersion kernel $\mathscr{D}_{ij}(x',x) = \mathscr{D}_{ji}(x,x')$, such that the variation of $\mathscr{D}$ with $x+x'$ is much slower than that with $x-x'$. The general approach is to seek solutions of (1) in the eikonal form $Z_i(x) = A_i(x)\exp[i\psi(x)]$, where both the amplitude $A_i(x)$ and the wave vector $k_\mu(x) = \partial\psi/\partial x^\mu$ are slowly varying objects. Then, to $O(\delta)$ ($\delta \ll 1$ being a small dimensionless parameter characterizing the slow space-time variation of the background) the problem can be reduced in some cases to the solution of coupled mode equations for just two components of the slow amplitude $A_i(x)$ (A_a and A_b, respectively):[4]

$$\begin{aligned} &D_a(k,x)A_a + \eta A_b = iL(D_a)A_a; \\ &\eta^* A_a + D_b(k,x)A_b = iL(D_b)A_b \,. \end{aligned} \tag{2}$$

The functions $D_{a,b}(k,x)$ in Eq. (2) are real, while the small $[O(\delta)]$ coupling parameter η may generally be complex. Furthermore, $D_{a,b}$ and η can be expressed algebraically in terms of the elements of the local dispersion matrix $D_{ij}(k,x) = \int d^4s \, \mathscr{D}_{ij}(x - s/2, x + s/2)\exp(-ik_\mu s^\mu)$. The differential operator L in (2) is of $O(\delta)$ and defined via $L(D)(\cdots) = (\partial D/\partial k_\mu)[\partial(\cdots)/\partial x^\mu] + \frac{1}{2}[d(\partial D/\partial k_\mu)/dx^\mu](\cdots)$. The physical meaning of the reduced system (2) is clear: when the coupling parameter η vanishes, the system reduces to the problem of propagation of two independent geometric optics modes for which the dispersion functions $D_{a,b}$ serve as the Hamiltonians in defining independent rays in the phase space $\{k,x\}$ along which $D_{a,b} \equiv 0$, while the decoupled equations (2) describe the slow variation of the amplitudes $A_{a,b}$ along the rays. If, in contrast, $\eta \neq 0$, Eqs. (2) comprise a more difficult problem of a conservative pairwise linear mode coupling, embedded in system (1), the solution of which in general geometry was obtained only recently.[3] Typically, the efficient mode interaction takes place in the vicinity of the crossing point (k_0,x_0) at which a ray of mode a in the eight-dimensional phase space (k,x) crosses the surface $D_b(k,x)=0$. Thus, $D_a = D_b = 0$ at the crossing point. If initially (prior to the crossing) only mode A_a is excited, then the mode A_b is generated in the vicinity of (k_0,x_0) and the asymptotic transmission/amplification coefficient of mode a after passing the crossing region is given by

$$T = \exp(\mp\pi|\eta|^2/|B|), \tag{3}$$

where the Poisson bracket $B = (\partial D_a/\partial k_\mu) \times(\partial D_b/\partial x^\mu) - (\partial D_b/\partial k_\mu)(\partial D_a/\partial x^\mu)$ is evaluated at the crossing point. The choice of the minus or plus sign in (3) depends on whether the two coupled modes have the same or different action density signs, respectively.[5] Linear mode conversion is a localized phenomenon, and the conversion efficiency decreases rapidly as one leaves the vicinity of the crossing point due to the growing mismatch between the

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phases of the coupled modes. This is the main reason, for example, for the finite value of the amplification coefficient in the case of the unstable coupling [the positive sign in (3)].

In this work we consider the effect of adding a weak nonlinearity to Eq. (1). It will be shown that the most important feature of this addition is the possibility of a substantial increase of the spatial width of the mode coupling region and the resulting significant modification of the transmission/amplification result (3). In the case of the unstable coupling, for example, T may become as large as allowed by the weak nonlinearity assumption. This increased conversion efficiency is achieved via the spatial autoresonance (SAR) effect, i.e., the automatic and continuous spatial adjustment (subject to certain conditions) of the nonlinear resonance condition between the coupled modes. The mechanism of the SAR comprises the spatial analog of the dynamic autoresonance effect studied previously in several applications.[6-8]

The scope of the present work is as follows. In Sec. II we describe the generalization of the congruent reduction technique,[4] yielding the weakly nonlinear coupled mode equations in general geometry. Sections III–V are devoted to the solution of the coupled mode equations for one-dimensional situations. We consider the case when only one of the coupled modes is launched toward the crossing point. The initial stage of the mode interaction under these conditions is investigated in Sec. III, showing the possibility of the characteristic trapping of the phase difference between the modes. Based on this initial trapping effect, the SAR theory for the stable conservative coupling is developed in Sec. IV and illustrated by numerical examples. Finally, the case of the unstable (positive–negative action density) mode coupling is investigated in Sec. V.

II. WEAKLY NONLINEAR COUPLED MODE EQUATIONS

Let us add the most simple *local* quadratic nonlinearity to (1), i.e., consider the system

$$\int d^4x'\, \mathscr{D}_{ij}(x,x')Z_j(x')+c_{ijk}(x)Z_j(x)Z_k(x)=0, \qquad (4)$$

where c_{ijk} are slowly varying real coefficients. Without loss of generality, we may assume that $c_{ijk}=c_{ikj}$. We shall also assume that $c_{ijk}=c_{jik}$ yielding, as will be shown below, small-amplitude action flux conservation by system (4). Equivalently, the symmetry of c_{ijk} with respect to the interchange of the indexes can be identified as the Manley–Row symmetry following from the variational principle for Eq. (4). In view of the nonlinearity, we generalize the usual eikonal representation by including higher harmonics, and seek solutions of (4) of the form

$$Z_i(x)=A_i^{(0)}(x)+\sum_{n=1}^{\infty}\mathrm{Re}\{A_i^{(n)}(x)\exp[in\psi(x)]\}, \qquad (5)$$

where $A_i^{(n)}$ and $k_\mu(x)=\partial\psi/\partial x^\mu$ are assumed to vary slowly with x. Suppose now that we excite only the term $n=1$ in (5) at the boundary of the region of interest, and that $|A_i^{(1)}|$ are *small* and formally of $O(\delta^{1/2})$ [alternatively, we could assume that $|A_i^{(1)}|$ are of $O(1)$, but c_{ijk} are of $O(\delta)$]. Then, $A_i^{(0)}$ and $A_i^{(2)}$ in (5) will be of $O(\delta)$ (see below), and we can neglect, to this order, all the terms with $n>2$. Next, we substitute this truncated form of (5) into (4), make the usual $O(\delta)$ geometric optics expansion with respect to x in the integrand,[9] and obtain the slow-amplitude transport equations:

$$D_{ij}^{(0)}A_j^{(0)}+\tfrac{1}{2}c_{ijk}A_j^{(1)}A_k^{(1)*}=0, \qquad (6)$$

$$(D_{ik}^{(1)}+2c_{ijk}A_j^{(0)})A_k^{(1)}+c_{ijk}A_j^{(2)}A_k^{(1)*}=iL^{(1)}(D_{ik}^{(1)})A_k^{(1)}, \qquad (7)$$

$$(D_{ik}^{(2)}+2c_{ijk}A_j^{(0)})A_k^{(2)}+c_{ijk}A_j^{(1)}A_k^{(1)}=iL^{(2)}(D_{ik}^{(2)})A_k^{(2)}, \qquad (8)$$

where $\mathbf{D}^{(n)}$ and the operator $L^{(n)}$ are defined as in Eq. (2), but with k replaced by nk. Now we see that Eqs. (6)–(8) yield the assumed ordering of $A_j^{(0)}$ and $A_j^{(2)}$. Indeed, to $O(\delta)$ and, assuming that matrices $D_{ij}^{(0)}$ and $D_{ij}^{(2)}$ are nonsingular, Eqs. (6) and (8) yield

$$A_j^{(0)}=d_{jmn}^{(0)}A_m^{(1)}A_n^{(1)*}, \quad A_j^{(2)}=2d_{jmn}^{(2)}A_m^{(1)}A_n^{(1)}, \qquad (9)$$

where $d_{jmn}^{(0,2)}=-\frac{1}{2}\mathsf{D}_{ji}^{(0,2)}c_{imn}$ with the sans serif denoting inverse matrices. These expressions substituted into Eq. (7) then lead to the weakly nonlinear transport equation for $A_j^{(1)}$:

$$D_{ij}^{(1)}A_j^{(1)}+f_{imnj}A_m^{(1)}A_n^{(1)*}A_j^{(1)}=iL^{(1)}(D_{ij}^{(1)})A_j^{(1)}, \qquad (10)$$

where

$$f_{imnj}=-(c_{ijk}\mathsf{D}_{kl}^{(0)}c_{lmn}+c_{ink}\mathsf{D}_{kl}^{(2)}c_{lmj}). \qquad (11)$$

We see that the assumed symmetry of c_{ijk} with respect to the interchange of the indices and the Hermiticity of $\mathbf{D}^{(0,2)}$ yield $f_{imnj}=f_{mijn}^{*}$ and thus Eq. (10) conserves the small-amplitude action flux associated with the main perturbation:

$$d_\mu J^\mu=0, \quad J^\mu=A_i^{(1)*}\left(\frac{\partial D_{ij}^{(1)}}{\partial k_\mu}\right)A_j^{(1)}, \qquad (12)$$

where $d_\mu=\partial/\partial x^\mu+(\partial k_\nu/\partial x^\nu)\partial/\partial k_\mu$. Note that the expression for the action flux J^μ in (12) is the lowest significant order result in terms of the wave amplitudes and all higher-order nonlinear contributions to the flux were neglected in the process of reduction of the transport equation (10).

At this point we apply the congruent reduction[4] ideas in order to extract possible pairwise couplings embedded in Eq. (10). The congruent reduction method was originally developed for linear problems, and here we extend the approach to weakly nonlinear situations. First, we linearly transform the field $\mathbf{Z}^{(1)}(x)=\mathbf{A}^{(1)}(x)\exp[i\psi(x)]$, i.e., introduce $\mathscr{Z}(x)=\mathscr{A}(x)\exp[i\psi(x)]$ via the invertible, weakly nonlocal transformation

$$Z_i^{(1)}(x)=\int d^4x'\, \mathscr{Q}_{ij}(x',x)\mathscr{Z}_j(x'), \qquad (13)$$

where $\mathscr{Q}$ is the transformation kernel, which, similarly to the dispersion kernel $\mathscr{D}$, varies rapidly with $x-x'$, but slowly with $x+x'$. Then, to $O(\delta)$, the slowly varying amplitudes $A^{(1)}$ and $\mathscr{A}$ are related via[4]

$$A_j^{(1)}=[Q_{j\beta}-iL(Q_{j\beta})]\mathscr{A}_\beta, \tag{14}$$

where the operator L is defined as in Eq. (2) and $\mathbf{Q}(k,x)$ is the local four-dimensional Fourier transform of $\mathscr{Q}$ [the definition of $\mathbf{Q}(k,x)$ is similar to that of the dispersion matrix $\mathbf{D}(k,x)$]. Next, we substitute (14) into (10) and operate by $\mathbf{Q}^\dagger-iL(\mathbf{Q}^\dagger)$ on the resulting equation from the left, yielding, to $O(\delta)$,

$$D'_{\alpha\beta}\mathscr{A}_\beta+f'_{\alpha\gamma\delta\beta}\mathscr{A}_\gamma\mathscr{A}^*_\delta\mathscr{A}_\beta=iL(D'_{\alpha\beta})\mathscr{A}_\beta, \tag{15}$$

where [compare to Eq. (14) in Ref. 4]

$$D'_{\alpha\beta}=Q^*_{\alpha i}D_{ij}Q_{j\beta}+(i/2)(\{Q^*_{\alpha i},D_{ij}\}Q_{j\beta}+Q^*_{\alpha i}\{D_{ij},Q_{j\beta}\}$$
$$+\{Q^*_{\alpha i},Q_{j\beta}\}D_{ij}). \tag{16}$$

The Poisson brackets in the last equation are defined as $\{a,b\}=(\partial a/\partial k_\mu)(\partial b/\partial x^\mu)-(\partial b/\partial k_\mu)(\partial a/\partial x^\mu)$ and

$$f'_{\alpha\gamma\delta\beta}=Q^*_{i\alpha}Q^*_{n\delta}f_{imnj}Q_{m\gamma}Q_{j\beta}.$$

Thus, the form of Eq. (15) for the transformed amplitude is the same as that of Eq. (10) for the original amplitude $\mathbf{A}^{(1)}$. Also, the matrix $D'_{\alpha\beta}$ is Hermitian and the coefficients $f'_{\alpha\gamma\delta\beta}$ have the same symmetry with respect to the interchange of indices as f_{imnj}. Finally,

$$J^\mu=A_i^{(1)*}\left(\frac{\partial D_{ij}^{(1)}}{\partial k_\mu}\right)A_j^{(1)}=\mathscr{A}^*_\alpha\left(\frac{\partial D'_{\alpha\beta}}{\partial k_\mu}\right)\mathscr{A}_\beta+O(\delta). \tag{17}$$

Now we choose the transformation $\mathbf{Q}$ so that a maximum number of the components of $\mathscr{A}$ become small [of $O(\delta^{3/2})$] as compared to the components of the original amplitude $\mathbf{A}$ [assumed to be of $O(\delta^{1/2})$]. Reference 4 describes a systematic step-by-step algorithm for finding such an annihilating transformation in linear problems. The same technique remains applicable for the nonlinear system (10), since the nonlinear terms in this system are of $O(\delta^{3/2})$ by assumption. Therefore, as in the linear theory,[4] if a *pairwise* mode coupling is embedded in (10), one can always find $\mathbf{Q}$ such that *all* components of $\mathscr{A}$ *but two* (say $\mathscr{A}_a$ and $\mathscr{A}_b$) are annihilated. Then, to $O(\delta)$, all the summations in (15) can be restricted to $\alpha,\beta=a,b$ only. This yields the desired second-order system of coupled mode equations of form (15) for the reduced amplitude $\mathscr{A}^r=(\mathscr{A}_a,\mathscr{A}_b)^T$ with $\mathbf{D}'$ replaced by the reduced (2×2) matrix

$$\mathbf{D}^r=\begin{bmatrix}D_a & \eta\\ \eta^* & D_b\end{bmatrix}+\Delta^r, \tag{18}$$

where we represent by Δ^r the $O(\delta)$ part with the Poisson brackets in Eq. (16). As mentioned above, conventional linear pairwise mode couplings take place in the vicinity of crossings (k_0,x_0) in the phase space at which $D_a(k_0,x_0)=D_b(k_0,x_0)=0$ and η is of $O(\delta)$. We can usually view $\mathbf{D}$ and $\mathbf{Q}$ in these regions as locally linear functions of k and x, so that Δ^r in (18) is a constant matrix of $O(\delta)$. Therefore, we can formally omit Δ^r in (18) by replacing $D_a\Rightarrow D_a+\Delta^r_{aa}$, $D_b\Rightarrow D_b+\Delta^r_{bb}$, and $\eta\Rightarrow\eta+\Delta^r_{ab}$. This replacement corresponds to a small translation of the crossing point in the phase space and a change in the value of the coupling coefficient η.

Finally, we make another important assumption related to the strength of the nonlinear coupling between $\mathscr{A}_a$ and $\mathscr{A}_b$. Typically, pairwise mode couplings in the linear theory require the existence of an additional small parameter in the problem, *assuring* the smallness of the coupling coefficient η over an extended region of the phase space (the normal degeneracy[4]). By analogy, we also assume the existence of such a *global* small parameter in our generalized problem, but restrict the theory to the cases when this parameter assures the smallness of *both* the linear and nonlinear coupling coefficients between the modes, i.e., when, in addition to η, all the coefficients $f_{\alpha\gamma\delta\beta}(\alpha,\beta=a,b)$ but $c_a=f'_{aaaa}$ and $c_b=f'_{bbbb}$ are of $O(\delta)$. Then the reduced coupled mode equations to the desired order become

$$\begin{aligned}&-i(D_a\mathscr{A}_a+c_a|\mathscr{A}_a|^2\mathscr{A}_a+\eta\mathscr{A}_b)\\&\quad=\left(\frac{\partial D_a}{\partial k_\mu}\right)\left(\frac{\partial\mathscr{A}_a}{\partial x^\mu}\right)+\frac{\mathscr{A}_a}{2}\frac{d(\partial D_a/\partial k_\mu)}{dx^\mu},\\&-i(D_b\mathscr{A}_b+c_b|\mathscr{A}_b|^2\mathscr{A}_b+\eta^*\mathscr{A}_a)\\&\quad=\left(\frac{\partial D_b}{\partial k_\mu}\right)\left(\frac{\partial\mathscr{A}_b}{\partial x^\mu}\right)+\frac{\mathscr{A}_b}{2}\frac{d(\partial D_b/\partial k_\mu)}{dx^\mu}.\end{aligned} \tag{19}$$

Note that because of the symmetries of $f_{\alpha\gamma\delta\beta}$ the coefficients $c_{a,b}$ in (19) are real. As a result, similar to the original system (15), the reduced equations (19) yield the conservation law

$$d_\mu J^\mu=0,\quad J^\mu=\left(\frac{\partial D_a}{\partial k_\mu}\right)|\mathscr{A}_a|^2+\left(\frac{\partial D_b}{\partial k_\mu}\right)|\mathscr{A}_b|^2. \tag{20}$$

III. INITIAL INTERACTION PHASE IN ONE-DIMENSIONAL COUPLINGS

In this section we start with the solution of the coupled mode equations (19). However, in order to reduce the complexity of the problem, we shall limit the analysis to the one-dimensional case. Thus, we assume that the dispersion functions $D_{a,b}$ vary in one spatial direction (say direction z) and, as a result, the components ω, k_x, and k_y of k_μ are constants, while $A_{a,b}$ and $k_z=\partial\psi/\partial z=\kappa$ are functions of z. In this case our coupled mode system (19) becomes

$$\begin{aligned}&-i(D_a\mathscr{A}_a+c_a|\mathscr{A}_a|^2\mathscr{A}_a+\eta\mathscr{A}_b)\\&\quad=\left(\frac{\partial D_a}{\partial\kappa}\right)\left(\frac{d\mathscr{A}_a}{dz}\right)+\frac{\mathscr{A}_a}{2}\frac{d(\partial D_a/\partial\kappa)}{dz},\\&-i(D_b\mathscr{A}_b+c_b|\mathscr{A}_b|^2\mathscr{A}_b+\eta^*\mathscr{A}_a)\\&\quad=\left(\frac{\partial D_b}{\partial\kappa}\right)\left(\frac{d\mathscr{A}_b}{dz}\right)+\frac{\mathscr{A}_b}{2}\frac{d(\partial D_b/\partial\kappa)}{dz},\end{aligned} \tag{21}$$

and (20) yields

$$J_z=V_aW_a+V_bW_b=\text{const}, \tag{22}$$

where

$$V_{a,b}=\frac{-(\partial D_{a,b}/\partial\kappa)}{(\partial D_{a,b}/\partial\omega)}, \quad W_{a,b}=\left(\frac{\partial D_{a,b}}{\partial\omega}\right)|A_{a,b}|^2, \tag{23}$$

are interpreted as the group velocity components in the direction of the inhomogeneity and action densities of the coupled modes.

Now consider the situation when asymptotically, far from the crossing point, the velocities V_a and V_b are both positive (the other possibilities can be treated similarly). Then we may encounter the following two cases. The first case corresponds to the situation when the action densities W_a and W_b of the modes both have the same sign (say positive, for definiteness, i.e., $\partial D_{a,b}/\partial\kappa<0$) and therefore J_z is positive definite. In this case (hereafter referred to as the *stable coupling*) the magnitudes $|\mathscr{A}_a|$ and $|\mathscr{A}_b|$ of the two coupled modes are bounded. The second possibility, in contrast, describes the situation when the action density signs of the two modes is different (say $W_a>0$ and $W_b<0$, i.e., $\partial D_a/\partial\kappa<0$ and $\partial D_b/\partial\kappa>0$). In this case, the constancy of J_z allows, in principle, an unlimited spatial growth of $|\mathscr{A}_a|$ and $|\mathscr{A}_b|$ as the modes copropagate in the direction of the inhomogeneity and, consequently, we shall use the term *unstable coupling* when referring to this case in the following. In order to see the difference between the above-mentioned two cases in the coupled mode system directly, we introduce new dimensionless dependent variables:

$$A_{a,b}=\mathscr{A}_{a,b}J_z^{-1}\sqrt{\left|\frac{\partial D_{a,b}}{\partial\kappa}\right|}, \tag{24}$$

for which (21) yields

$$\begin{aligned}&\frac{dA_a}{dz}-i(D_a'+c_a'|A_a|^2)A_a=i\eta' A_b,\\ &\frac{dA_b}{dz}\mp i(D_b'+c_b'|A_b|^2)A_b=\pm i\eta'^*A_a,\end{aligned} \tag{25}$$

where the upper and lower signs in the second equation correspond to the stable and unstable couplings, respectively, and

$$D_{a,b}'=D_{a,b}\left|\frac{\partial D_{a,b}}{\partial\kappa}\right|^{-1};$$

$$c_{a,b}'=c_{a,b}J_z\left|\frac{\partial D_{a,b}}{\partial\kappa}\right|^{-2};$$

$$\eta'=\eta\left|\left(\frac{\partial D_a}{\partial\kappa}\right)\left(\frac{\partial D_b}{\partial\kappa}\right)\right|^{-1/2}.$$

The flux conservation law (22) in these new variables becomes

$$|A_a|^2\pm|A_b|^2=1. \tag{26}$$

As an additional preliminary step, we convert (25) into a system of real equations:

$$\begin{aligned}&\frac{dB_a}{dz}=-|\eta'|B_b\sin(\phi_b-\phi_a+\theta),\\ &\frac{dB_b}{dz}=\pm|\eta'|B_a\sin(\phi_b-\phi_a+\theta),\\ &\frac{d\phi_a}{dz}=D_a'+c_a'B_a^2+|\eta'|\left(\frac{B_b}{B_a}\right)\cos(\phi_b-\phi_a+\theta),\\ &\frac{d\phi_b}{dz}=\pm\left[D_b'+c_b'B_b^2+|\eta'|\left(\frac{B_a}{B_b}\right)\cos(\phi_b-\phi_a+\theta)\right],\end{aligned} \tag{27}$$

where the magnitudes and imaginary phases of various quantities are defined via

$$A_{a,b}=B_{a,b}\exp(i\phi_{a,b}); \quad \eta'=|\eta'|\exp(i\theta). \tag{28}$$

Equations (27) show that only the phase difference $\phi=\phi_b-\phi_a+\theta$ is needed in order to describe the magnitudes $B_{a,b}$; the three quantities ($B_{a,b}$ and ϕ) are governed by

$$\begin{aligned}&\frac{dB_a}{dz}=-|\eta'|B_b\sin\phi,\\ &\frac{dB_b}{dz}=\pm|\eta'|B_a\sin\phi,\\ &\frac{d\phi}{dz}=\pm D_b'-D_a'\pm c_b'B_b^2-c_a'B_a^2+|\eta'|[\pm(B_a/B_b)\\ &\qquad-(B_b/B_a)]\cos\phi.\end{aligned} \tag{29}$$

At this point we proceed to the detailed analysis of the situation when initially, at $z=-\infty$, one excites mode a only, i.e., $B_a(-\infty)=1$, while $B_b(-\infty)=0$. We define the *slowly* varying wave vector $\kappa(z)$ via the usual local dispersion relation of the linear problem

$$D_a'[\kappa(z),z]=0, \tag{30}$$

and, as in the linear mode conversion case, assume that the effective interaction between the modes takes place in the vicinity of the crossing point (κ_0,z_0) at which $D_a'(\kappa_0,z_0) = D_b'(\kappa_0,z_0) = 0$. Consequently, we expand D_a' and D_b' in the vicinity of (κ_0,z_0) and keep only the linear terms in this expansion. Then (30) yields

$$\kappa-\kappa_0=-\frac{\partial D_a'/\partial z}{\partial D_a'/\partial\kappa}(z-z_0), \tag{31}$$

and, therefore,

$$D_b'=\left(\frac{\partial D_b'}{\partial\kappa}\right)(\kappa-\kappa_0)+\left(\frac{\partial D_b'}{\partial z}\right)(z-z_0)=C(z-z_0), \tag{32}$$

where $C = B/(\partial D_a'/\partial\kappa)$, $B = (\partial D_a'/\partial\kappa)(\partial D_b'/\partial z) - (\partial D_b'/\partial\kappa)(\partial D_a'/\partial z)$, and all the partial derivatives in (31) and (32) are evaluated at the crossing point. Finally, assuming $C>0$ for definiteness, and for simplicity that $c_{a,b}'$ are constants, we introduce $\phi_\pm=\pm\phi$, define the dimensionless coordinate, coupling constant, and nonlinearity parameters

$$\tau = C^{1/2}(z - z_0) \pm \beta_a;$$

$$\mu = C^{-1/2}|\eta'|; \tag{33}$$

$$\beta_{a,b} = -C^{-1/2}c'_{a,b},$$

and rewrite (29) as

$$\frac{dB_a}{d\tau} = \mp \mu B_b \sin \phi_\pm ,$$

$$\frac{dB_b}{d\tau} = \mu B_a \sin \phi_\pm , \tag{34}$$

$$\frac{d\phi_\pm}{d\tau} = \tau - \beta B_b^2 - \mu \left[\pm \left(\frac{B_b}{B_a}\right) - \left(\frac{B_a}{B_b}\right) \right] \cos \phi_\pm ,$$

where $\beta = \beta_b + \beta_a$. This is the desired one-dimensional final form of the coupled mode equations.

We proceed to the solution of (34) subject to the above-mentioned "ideal" boundary conditions when only mode a is excited at $\tau = -\infty$. One limiting case of (34) was already studied previously. Indeed, if one defines the parameter $\sigma \equiv \mu/\beta$, then for $|\sigma| \gg 1$ we can neglect the term βB_b^2 in the third equation in (34) and use the approximate system

$$\frac{dB_a}{d\tau} = \mp \mu B_b \sin \phi_\pm ,$$

$$\frac{dB_b}{d\tau} = \mu B_a \sin \phi_\pm , \tag{35}$$

$$\frac{d\phi_\pm}{d\tau} = \tau - \mu \left[\pm \left(\frac{B_b}{B_a}\right) - \left(\frac{B_a}{B_b}\right) \right] \cos \phi_\pm .$$

But βB_b^2 is the only term in (34) due to the nonlinearity in the original system (4). Therefore, the $|\sigma| \gg 1$ limit corresponds to the *linear* mode conversion problem, the solution of which is known and was described in the Introduction.

Suppose now that one encounters the opposite limit, i.e.,

$$|\sigma| = \mu/|\beta| \ll 1. \tag{36}$$

This case yields under certain conditions the SAR solution (see Secs. IV and V) and comprises the subject of our discussion in the rest of this work. Despite the strong inequality in (36), the nonlinear term βB_b^2 vanishes at $-\infty$ due to the boundary conditions and remains small as long as B_b is small. Therefore, in the *initial interaction phase*, when $|\beta| B_b^2 \ll \mu/B_b$, or (using an order of magnitude estimate)

$$B_b \ll \tfrac{1}{2}|\sigma|^{1/3}, \tag{37}$$

the linear mode conversion equations (35) comprise a good approximation. The remainder of this section treats this initial interaction phase in detail as a necessary preparation for the complete solution of the problem in the *main interaction phase*, where the inequality (37) is reversed.

The first equation in (34) shows that B_a remains almost constant in the initial interaction phase and, therefore, in the last two equations in (34), we can set $B_a = 1$ and neglect the small terms βB_b^2 and $\mu B_b/B_a \cos \phi_\pm$. This results in the following approximate system:

$$\frac{dB_b}{d\tau} = \mu \sin \phi_\pm ,$$

$$\frac{d\phi_\pm}{d\tau} = \tau + \left(\frac{\mu}{B_b}\right) \cos \phi_\pm . \tag{38}$$

Equations (38) are equivalent to a single complex equation

$$\frac{dR}{d\tau} - i\tau R = i\mu, \tag{39}$$

where $R \equiv B_b \exp(i\phi_\pm)$. The solution of (39), subject to $R(-\infty) = 0$, is

$$R(\tau) = \pm i\mu \exp\left(\frac{\tau^2}{2}\right) \int_{-\infty}^{\tau} d\tau' \exp\left(\frac{-\tau'^2}{2}\right), \tag{40}$$

or, by expressing the integral on the right-hand side in the last equation via the auxiliary functions $f(z)$ and $g(z)$ associated with Fresnel integrals $C(z)$ and $S(z)$,[10]

$$R = B_b \exp(i\phi_\pm) = \mu \sqrt{\pi} [f(|\tau|/\sqrt{\pi}) + ig(|\tau|/\sqrt{\pi})]. \tag{41}$$

Asymptotically, for large arguments, $f(x) \sim 1/\pi x$ and $g(x) \sim 1/\pi^2 x^3$ and, therefore, when $|\tau|$ is large enough,

$$B_b \approx \mu/|\tau|, \tag{42}$$

and $\phi_\pm$ is approximately zero. This initial positioning of the phase difference $\phi_\pm$ at 0 is the important property of the coupled mode interaction, leading (see the next section) to the SAR phenomenon when one adds a sufficiently strong nonlinearity.

Approximation (42) and our conclusion on the phase difference positioning at zero are valid only for $f \gg g$ and therefore (again using an order of magnitude estimate) we require $|\tau| > 3$ in (42), since $g(3/\sqrt{\pi}) \approx 0.1 f(3/\sqrt{\pi})$.[11] On the other hand, the use of (42) in (37) yields $|\tau| > 2\mu|\sigma|^{-1/3}$. Therefore, we define $\tau_{\text{init}} \equiv -\max(3, 2\mu|\sigma|^{-1/3})$ as the end point of the initial interaction phase.

The final question to be considered in this section is whether the phase difference positioning at 0 prevails if B_b does not vanish at the boundary of the region of interest. Suppose that the initial point of integration is at $\tau = \tau_0 < \tau_{\text{init}}$. Then, for a nonvanishing $R(\tau_0) = B_b(\tau_0)\exp[i\phi_\pm(\tau_0)]$, instead of Eq. (41), we have

$$B_b \exp(i\phi_\pm) = B_b(\tau_0)\exp[i\phi_\pm(\tau_0) + i(\tau^2 - \tau_0^2)/2] + \mu\sqrt{\pi}[(f+ig)_{|\tau|/\sqrt{\pi}} - i(f + ig)_{|\tau_0|/\sqrt{\pi}} \exp[i(\tau^2 - \tau_0^2)/2], \tag{43}$$

or, for $|\tau_0| \gg |\tau| > |\tau_{\text{init}}|$,

$$B_b e^{i\phi_\pm(\tau)} = [B_b(\tau_0) e^{i\phi_\pm(\tau_0)} - \mu/|\tau_0|] e^{i(\tau^2 - \tau_0^2)/2} + \mu/|\tau|. \tag{44}$$

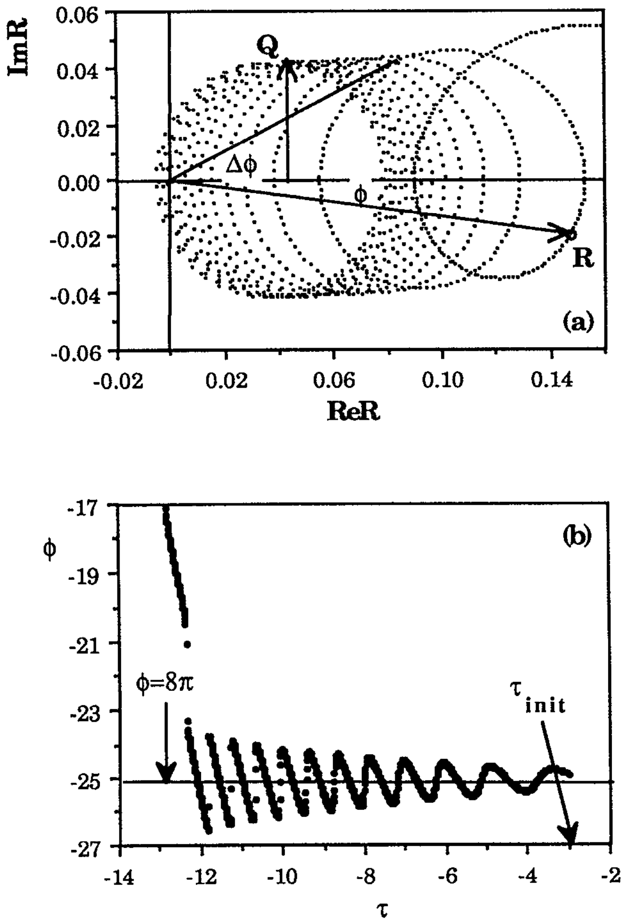

FIG. 1. Initial interaction phase in one-dimensional couplings. (a) The evolution of the solution $R=B_b\exp(i\phi_\pm)$ of Eq. (39) in the complex R plane for $\mu=0.5$, $\tau_0=-14$, $\phi_\pm(\tau_0)=0$, and $R(\tau_0)=0.075$; (b) the corresponding dependence of the phase difference $\phi_\pm$ on τ.

By denoting the constant object in the square brackets in (44) by $Q\exp(i\Psi_0)$, with Q and Ψ_0 being the absolute value and the imaginary phase, we finally have

$$B_b e^{i\phi_\pm(\tau)}=Qe^{i[\Psi_0+(\tau^2-\tau_0^2)/2]}+\mu/|\tau|. \tag{45}$$

Thus, the function $R(\tau)=B_b(\tau)\exp[i\phi_\pm(\tau)]$ in the initial interaction phase can be viewed as a point in the complex R plane moving uniformly with τ^2 on a circle of constant radius Q, with the center of the circle positioned at $\mu/|\tau|$ and therefore shifting in the positive Re R direction with a decrease of $|\tau|$. This geometric interpretation is illustrated in Fig. 1(a), showing the evolution of R in the complex R plane as found by solving Eq. (39) for the case $\mu=0.5$, $\tau_0=-14$, and $R(\tau_0)=0.075$. The corresponding dependence of $\phi_\pm$ on τ is shown in Fig. 1(b). We see that $\phi_\pm$ decreases monotonically as $|t|$ decreases as long as $Q>\mu/|\tau|$. For $Q<\mu/|\tau|$, in contrast, $\phi_\pm(\text{mod } 2\pi)$ oscillates around 0 and the amplitude $\Delta\phi$ of these oscillations can be estimated from $\sin\Delta\phi=Q\tau/\mu$, i.e., $\Delta\phi$ does not exceed $\pi/2$. Thus, the phase difference $\phi_\pm$ is *trapped* in the $Q<\mu/|\tau|$ case. We shall show in the next section that the SAR phenomenon is stable with respect to these characteristic *initial* oscillations of $\phi_\pm$. The proof of this statement will be given for the case of the *strong* initial trapping, i.e., when at τ_{init}, $\Delta\phi<\pi/6$. The mathematical condition for entering the strong trapping regime at the endpoint $\tau=\tau_{\text{init}}$ of the initial interaction phase (see the definition above) is

$$Q<\mu/2|\tau_{\text{init}}|. \tag{46}$$

If, at τ_0, B_b is of the same order as $\mu/|\tau_0|$, condition (46) is satisfied automatically. If, in contrast, $B_b(\tau_0)\gg\mu/|\tau_0|$, we can replace Q by $B_b(\tau_0)$ in the strong trapping condition (46). This completes our discussion of the initial interaction phase and we proceed to the region $\tau>\tau_{\text{init}}$, where a significant flux redistribution may take place and one cannot assume the approximate constancy of the magnitude of one of the modes, nor can one neglect the nonlinear term βB_b^2 in Eq. (34).

IV. SPATIAL AUTORESONANCE FOR STABLE COUPLING

At this point we return to the problem of solution of the original normalized coupled mode equations (34) in the stable-coupling case (the upper sign in the equations) for $\tau>\tau_{\text{init}}$. The analysis is simplified by revealing the Hamiltonian structure of (34). We recall that $\phi_+=\phi$ and $B_a^2+B_b^2=1$ in the case of interest, and introduce the new dependent variable

$$I=(B_a^2-B_b^2)/2\beta \tag{47}$$

in terms of which $B_a^2=\frac{1}{2}+\beta I$ and $B_b^2=\frac{1}{2}-\beta I$. Then (34) can be replaced by the equivalent system

$$\begin{aligned}\frac{dI}{dt}&=-\sigma X(I)\sin\phi,\\ \frac{d\phi}{dt}&=t+\beta^2 I-\sigma\left(\frac{dX}{dI}\right)\cos\phi,\end{aligned} \tag{48}$$

where $X(I)=(1-4\beta^2 I^2)^{1/2}=2B_aB_b$ and $t=\tau-\beta/2$. We see that I and ϕ are the canonical pair associated with the Hamiltonian

$$H(I,\phi,t)=tI+\beta^2I^2/2-\sigma X(I)\cos\phi. \tag{49}$$

Suppose now that the phase difference ϕ is strongly trapped in the vicinity of $t_{\text{init}}=\tau_{\text{init}}-\beta/2$, i.e., $\phi(\text{mod } 2\pi)$ oscillates around zero with the amplitude that is less than $\pi/6$ (see the discussion in the previous section). Assume also that the strong trapping of ϕ is preserved for $t>t_{\text{init}}$ (the assumption to be checked later). Then (49) can be approximated by

$$H(I,\phi,t)=tI+\beta^2I^2/2-\sigma X+\sigma X\phi^2/2. \tag{50}$$

Our goal is to show that under certain conditions (50) yields the possibility of *spatial autoresonance*, i.e., the automatic satisfaction of the *nonlinear* resonance condition

$$t+\beta^2 I+\sigma\frac{dX}{dI}\approx 0, \tag{51}$$

where the departure from zero is small [of $O(|\sigma|^{1/2})$] and has a rapid oscillatory nature. The variable I, at the same time, has a slowly varying "evolutionary" part and a superimposed small $[O(|\sigma|^{1/2})]$ oscillating component. If (51) is satisfied, the phase ϕ remains nearly constant on average, as follows from the second equation in (48). Then the first equation in (48) yields a slow monotonic variation of the evolutionary part of I, meaning a continuous action flux redistribution between the coupled modes over distances much longer than those predicted by the linear theory. The detailed proof of these statements is given below.

We seek solutions for I and ϕ of the form

$$I=I_0+\delta I, \quad \phi=\phi_0+\delta\phi,$$

where $I_0(t)$ and $\phi_0(t)$ are slowly varying functions (in the sense to be clarified later) of t, while δI and $\delta\phi$ oscillate rapidly. We also assume that the amplitude ΔI of the oscillating component δI is small ($\Delta I/I_0 \ll 1$), while ϕ_0 itself is small ($|\phi_0| \ll 1$), but the amplitude $\Delta\phi$ of the oscillations $\delta\phi$ is not necessarily much less than $|\phi_0|$. The canonical transformation $(I,\phi)\rightarrow(\delta I,\delta\phi)$ can be conveniently carried out by using the mixed variables generating function

$$\mathscr{G}(\phi,\delta I,t)=[\phi-\phi_0(t)][I_0(t)+\delta I]-\int \mathscr{f}(t)dt, \tag{52}$$

where I_0, ϕ_0, and $\mathscr{f}$ are selected so that one obtains the desired solution for the transformed set of variables. The proper selection is via

$$\frac{dI_0}{dt}=-\sigma X_0\phi_0, \qquad \frac{d\phi_0}{dt}=t+F(I_0), \tag{53}$$

and

$$\mathscr{f}=t+\beta^2I_0^2/2-\sigma X_0, \tag{54}$$

where $F(I_0)=\beta^2I_0-\sigma X_0'$ and X_0, X_0', and later X_0'' are X, and its first and second derivatives evaluated at I_0. The transformed Hamiltonian to second order in the transformed variables then assumes the simple form

$$\mathscr{H}(\delta I,\delta\phi,t)=H(I,\phi,t)+\frac{\partial\mathscr{G}}{\partial t}=F'(I_0)\frac{\delta I^2}{2}+\frac{\sigma X_0\,\delta\phi^2}{2}, \tag{55}$$

where $F'=dF/dI_0$.

Formally, $\mathscr{H}$ describes a linear pendulum with varying parameters. We are interested in finding the conditions when this variation is *adiabatic*, in the sense that if one defines the "frequency" Ω via

$$\Omega^2=\sigma X_0/F'(I_0), \tag{56}$$

then the relative change in I_0 (and therefore also in Ω) during the characteristic "period" $2\pi/\Omega$ is small, i.e.,

$$\left(\frac{2\pi}{\Omega}\right)\left|\frac{d(\ln I_0)}{dt}\right| \ll 1. \tag{57}$$

Such conditions can be found by analyzing Eqs. (53). Indeed, by differentiating the second equation in (53) and using the first equation, we obtain

$$\frac{d^2\phi_0}{dt^2}=1-\sigma X_0F'\phi_0. \tag{58}$$

If $\sigma X_0F'>0$, Eq. (58) describes a stable linear pendulum under the action of a unit normalized "torque." The desired *small* and *slowly varying* solution for ϕ_0 corresponds to the adiabatic equilibrium of this pendulum,

$$\phi_0=1/\sigma X_0F', \tag{59}$$

under the condition

$$|\sigma X_0F'|=\mu|\beta(X_0+4\sigma/X_0^2)| \gg 1, \tag{60}$$

which can be satisfied for all X_0 only if $\sigma>0$ and

$$3\sigma^{1/3}\mu\beta \gg 1. \tag{61}$$

Finally, the substitution of (59) into the first equation in (54) gives

$$\frac{dI_0}{dt}=-\frac{1}{F'}, \tag{62}$$

yielding

$$t+F(I_0)=0, \tag{63}$$

for $t>t_{\text{init}}$ if this equation is satisfied at t_{init}, which, according to (42), is the case, since $F(I_0)\approx -\sigma X_0'\approx \mu/B_b$ at the end point of the initial interaction phase.

Equation (63) is the slowly varying part of Eq. (51) and represents the analytic manifestation of the spatial nonlinear autoresonance in the system. It shows that the function $F(I_0)$ varies linearly with t, *regardless* of the rate of spatial variation of the linear dispersion functions D_a and D_b, as long as the inequality (61) (hereafter referred to as the *autoresonance condition*) is satisfied. The spatial variation of I_0 resulting from (63) then describes the action flux *redistribution* between the modes.

The graphical method comprises a convenient way for investigating possible solutions of (63) for I_0. This is illustrated in Fig. 2, showing the dependence of $-F$ on $2|\beta|I_0$ for $\mu=0.5$ and two values of $\beta=\pm 15$. We see in the figure that only for $\beta>0$ (the solid line), $-F$ is a monotonically decreasing function of $2|\beta|I_0$, and one can satisfy (63) continuously as t increases in the region $t>t_{\text{init}}$. The path of the evolution in this case is shown by arrows on the solid line in the Fig. 2. We see that asymptotically, for large t, $I_0\rightarrow -1/2\beta$ and therefore $B_a\rightarrow 0$, while $B_b\rightarrow 1$, meaning a complete transfer of the action flux from mode a to mode b. For small values of B_a, however, one must switch the treatment to that similar to what is described in Sec. III for small values of B_b. The role of the modes in the equations is simply interchanged in the final phase of the interaction, where the *detrapping* of ϕ takes place and the efficient mode coupling discontinues. We shall not go into the details of the analysis of this phase.

In the case $\beta<0$ (the dashed line in Fig. 2), the function $-F$ has a maximum at I_{0m}, given by

$$I_{0m}=-\frac{[1-(4|\sigma|)^{2/3}]^{1/2}}{2|\beta|}, \tag{64}$$

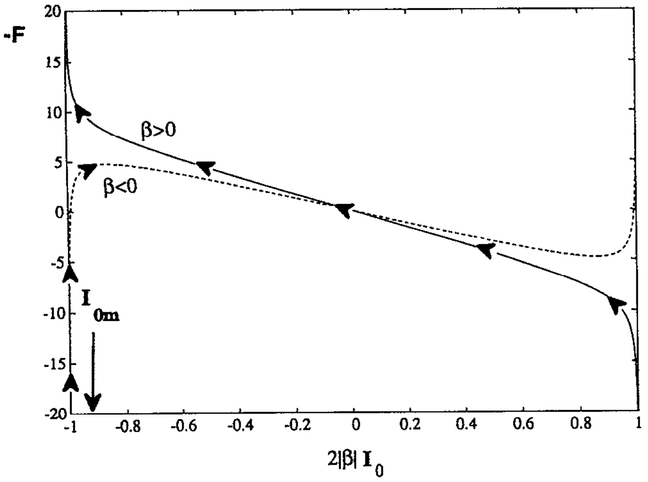

FIG. 2. The dependence of the function $-F$ (characterizing the stable coupling case) on $2|\beta|I_0$ for $\mu=0.5$ and $\beta=15$ (the solid line) and $\beta=-15$ (the dashed line). The arrows show the evolution path in the autoresonance regime.

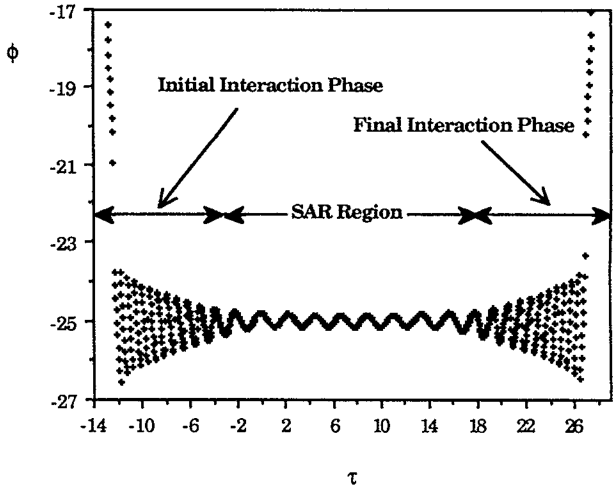

FIG. 3. The evolution of the phase difference ϕ in the stable coupling case for $\mu=0.5$, $\beta_a=15$, $\beta_b=0$, and the boundary conditions $B_a(\tau_0)=0.997$, $B_b(\tau_0)=0.077$, and $\phi(\tau_0)=0$ at $\tau_0=-14$.

[assuming, of course, $(4|\sigma|)^{2/3}<1$] and, therefore, one cannot satisfy (63) *continuously* by increasing t beyond the value $t_{\max}=-F(I_{0m})$. Also, $F'(I_{0m})=0$, and thus the autoresonance condition (60) is violated at I_{0m}. As the result, no efficient action flux redistribution between the modes takes place beyond $t_{\max}$. Finally, for $(4|\sigma|)^{2/3}\ll 1$, I_{0m} does not differ significantly from its initial value $\approx -\frac{1}{2}|\beta|$, i.e., the maximum value of B_b remains small after the interaction in this case.

In order to complete the picture, we must also characterize the evolution of the oscillating components $\delta\phi$ and δI during the autoresonance phase of the interaction. These two quantities are described by the Hamiltonian (55), i.e., their dynamics is formally that of a linear pendulum with adiabatically varying parameters. The adiabatic frequency of this pendulum is given in Eq. (56), and it only remains to analyze the slow evolution of the amplitudes $\Delta\phi$ and ΔI of the oscillations. We observe that (55) yields the adiabatic invariant[12]

$$\mathscr{I}=\frac{\mathscr{H}}{\Omega}=\frac{F'(\Delta I)^2}{2\Omega}=\frac{\sigma X_0(\Delta\phi)^2}{2\Omega}. \qquad (65)$$

Therefore,

$$\Delta\phi\sim(\sigma X_0 F')^{-1/2} \qquad (66)$$

and

$$\Delta I=(\sigma X_0/F')^{1/2}\,\Delta\phi. \qquad (67)$$

Thus, ΔI is indeed of $O(\sigma^{1/2})$ as assumed. Also with the increases of t, the amplitude $\Delta\phi$ decreases reaching the minimum value at $I_0=0$, i.e., when $B_{a0}=B_{b0}=1/\sqrt{2}$. With a further increase of t, the roles of the modes in the equations are formally interchanged and $\Delta\phi$ increases, gradually reaching the same value as at $t=t_{\text{init}}$ when B_{a0} becomes as small as B_{b0} at t_{init}. This proves our strong trapping assumption in the autoresonance phase of the interaction.

Now we illustrate the theory by numerical examples. Figures 3 and 4 show the results of the numerical solution of (34) for ϕ, $B_a=(\frac{1}{2}+\beta I)^{1/2}$ and $B_b=(\frac{1}{2}-\beta I)^{1/2}$ as functions of τ for the stable coupling case, subject to $B_a=0.997$, $B_b=0.077$, and $\phi=0$ at $\tau_0=-14$ and the parameters $\mu=0.5$, $\beta_a=15$, and $\beta_b=0$. For comparison, Fig. 5 shows the evolution of $B_{a,b}$ in the same example, but with $\beta_a=0$, i.e., the linear mode conversion case. We see that, as predicted, and in contrast to the linear case, the action flux transfer from mode a to mode b with nonlinearities included is almost complete. At the same time, the width $\Delta\tau\sim\beta_a$ of the efficient mode coupling region in the nonlinear case is much broader than in the linear mode conversion case, where $\Delta\tau\sim 1$. Figure 3 also shows the initial trapping stage followed by oscillations of ϕ, characteristic of the autoresonance. The frequency of these oscillations (also seen in Fig. 4 for $B_{a,b}$) is that predicted by Eq. (56), and the amplitude $\Delta\phi$ of the oscillations is at its minimum when $B_a=B_b$, as expected.

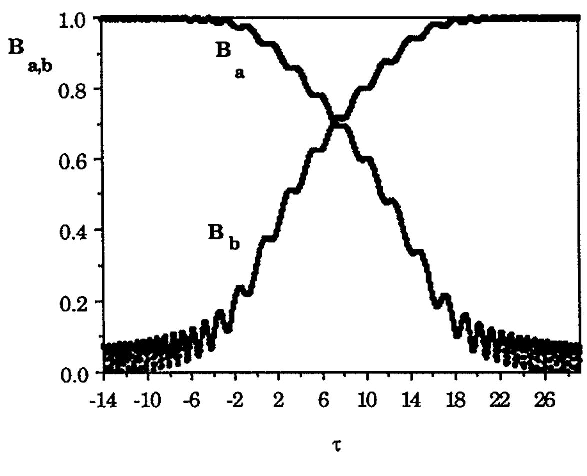

FIG. 4. The evolution of the magnitudes of the coupled modes $B_{a,b}$ in the stable coupling example shown in Fig. 3.

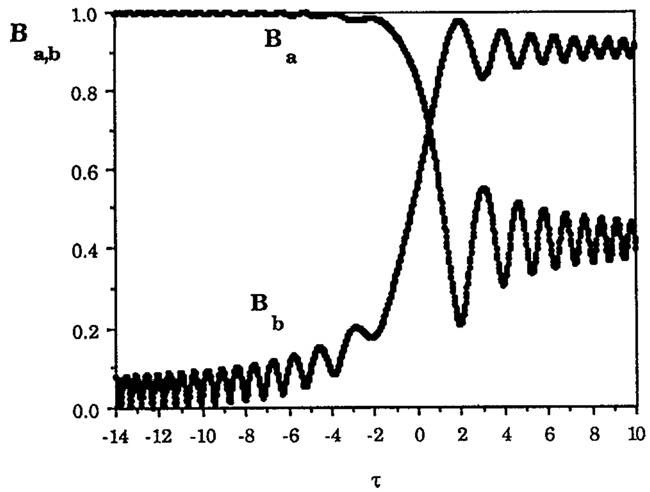

FIG. 5. The evolution of the magnitudes of the coupled modes $B_{a,b}$ in the stable linear mode coupling case ($\beta_a=\beta_b=0$). All the other parameters and the initial conditions are as in Figs. 3 and 4.

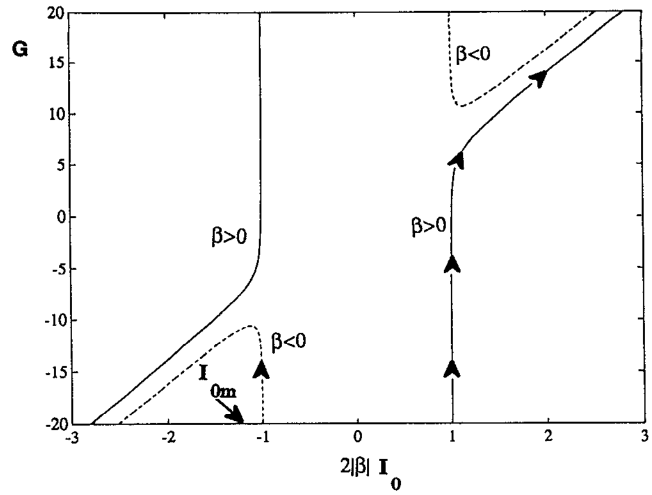

FIG. 6. The dependence of the function $-G$ (characterizing the unstable coupling case) on $2|\beta|I_0$ for $\mu=0.5$ and $\beta=15$ (the solid line) and $\beta=-15$ (the dashed line). The arrows show the evolution path in the autoresonance regime.

V. UNSTABLE CONSERVATIVE COUPLING

Consider now the unstable coupling case, described by system (34) with the lower signs in the equations. Since $B_a^2-B_b^2=1$ in this case, the convenient choice of the canonical variable I is now

$$I=(B_a^2+B_b^2)/2\beta. \tag{68}$$

Then, instead of (48) we have

$$\frac{dI}{dt}=\sigma Y(I)\sin\phi_-,$$
$$\frac{d\phi_-}{dt}=t-\beta^2 I+\sigma\left(\frac{dY}{dI}\right)\cos\phi_-, \tag{69}$$

where $t=\tau+\beta_a/2$ and $Y(I)=(4\beta^2I^2-1)^{1/2}$. The Hamiltonian characterizing the unstable coupling is, therefore,

$$H=tI-\beta^2I^2/2+\sigma Y(I)\cos\phi_-. \tag{70}$$

The discussion in Sec. III shows that the initial interaction phases of the stable *and* unstable couplings proceed similarly. Therefore, making again the strong trapping assumption for $t>t_{\text{init}}=\tau_{\text{init}}+\beta_a/2$, we arrive at the approximate Hamiltonian

$$H=tI-\beta^2I^2/2+\sigma Y-\sigma Y\phi_-^2/2. \tag{71}$$

The analysis of the dynamics associated with this Hamiltonian is similar to that of the previous section and also yields the possibility of the SAR regime, but now I_0 and ϕ_{-0} obey [compare to (59) and (63)]

$$\phi_{-0}=-1/\sigma Y_0G' \tag{72}$$

and

$$t+G(I_0)=0, \tag{73}$$

where $G(I_0)=-\beta^2I_0+\sigma Y_0'$, index 0 denotes the evaluation at I_0, and $Y_0'=dY_0/dI_0$. The frequency of the fast scale oscillations in the unstable coupling case is [compare to (56)]

$$\Omega^2=\sigma Y_0/G'(I_0). \tag{74}$$

As in Sec. IV, we require $\Omega^2>0$ and

$$|\sigma Y_0G'|=\mu|\beta(Y_0+4\sigma/Y_0^2)|\gg 1, \tag{75}$$

as the necessary autoresonance conditions. The inequality (75) then yields the same condition (61) as in the stable coupling case. In order to analyze solutions of (73) we present the function $-G$ vs $2|\beta|I_0$ in Fig. 6 for the same parameters as in Fig. 2 ($\mu=0.5$ and $\beta=\pm 15$). We see again that only in the case $\beta>0$ (the solid line in Fig. 6), $-G$ is a monotonically increasing function, which allows a continuous satisfaction of (73) with increase of t. However, in contrast to the stable coupling case, $-G$ is not bounded and, therefore, the autoresonance can, in principle, proceed indefinitely (see the arrows on the solid line in Fig. 6 showing the path of the evolution). The corresponding increase of I_0 means the amplification of both B_a and B_b to significantly larger values than those predicted by the linear theory. The only limitation in this case is a possible violation of the weak nonlinearity assumption at high amplification. In the case $\beta<0$, one reaches the maximum value of $-G$ at I_{0m} (see the arrows on the dashed line in Fig. 6 corresponding to the negative β case), beyond which (73) cannot be satisfied continuously with the increase of t. The analysis of the oscillating parts of ϕ_- and I in the unstable coupling case is similar to that in Sec. IV. In particular, $\delta\phi_-$ and δI can be expressed again in terms of the dynamics of the linear pendulum characterized by the Hamiltonian of form (55), with F' and X_0 replaced by G' and Y_0, respectively. The frequency of the oscillations predicted by this Hamiltonian is given by Eq. (74). As before, the new Hamiltonian yields the adiabatic invariant of form (65) (with G' and Y_0 replacing again F' and X_0), which can be used in finding the slowly varying amplitude of the fast oscillations $\delta\phi$ and δI. We shall skip the details of this

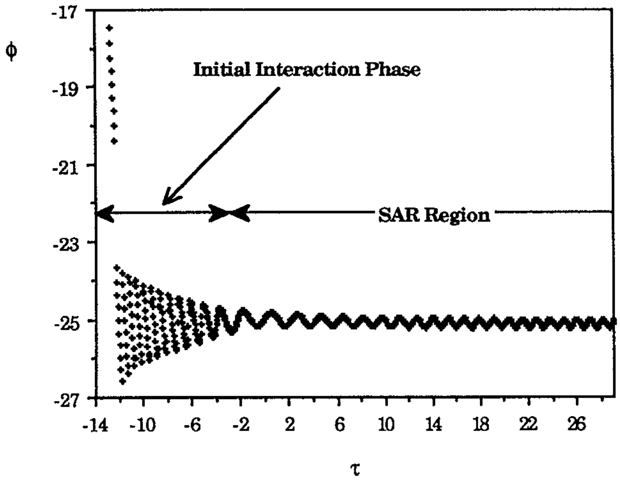

FIG. 7. The evolution of the phase difference ϕ in the unstable coupling case for $\mu=0.5$, $\beta_a=15$, $\beta_b=0$, and the boundary conditions $B_a(\tau_0)=1.003$, $B_b(\tau_0)=0.077$, and $\phi(\tau_0)=0$ at $\tau_0=-14$.

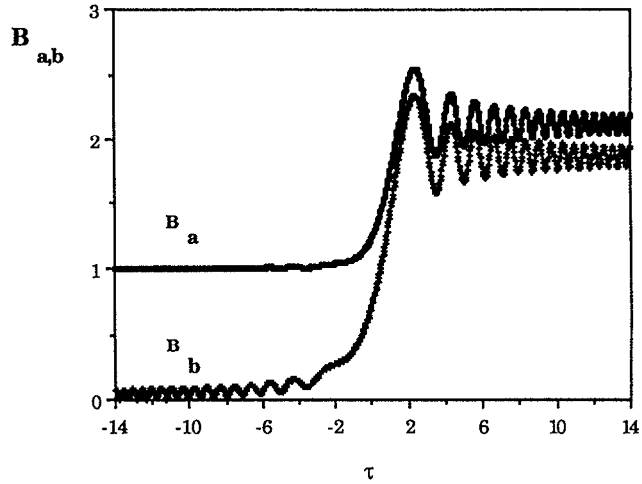

FIG. 9. The evolution of the magnitudes of the coupled modes $B_{a,b}$ in the unstable linear mode coupling case ($\beta_a=\beta_b=0$). All the other parameters and the initial conditions are as in Figs. 7 and 8. The amplification remains finite in this case.

analysis because of the similarity with that of Sec. IV, and proceed directly to numerical examples.

Figures 7 and 8 show the results of the numerical solutions of (34) for the same initial integration point, initial value of ϕ and parameters as in Figs. 3 and 4 [$\tau_0=-25$, $\phi(\tau_0)=0$, $\beta_a=15$, and $\beta_b=0$], but for $B_a=1.003$, $B_b=0.077$, and the choice of the lower signs in the equations corresponding to the unstable coupling. The same example, but with $\beta_a=0$, i.e., the linear mode conversion case, is shown in Fig. 9, for comparison. We see that the two cases proceed similarly in the initial interaction phase. Later, however, the inclusion of the nonlinearity leads to the SAR phenomenon, characterized by a continuous growth of the wave amplitudes, while in the linear case the interaction leads to finite amplification of the coupled modes as given by (3).

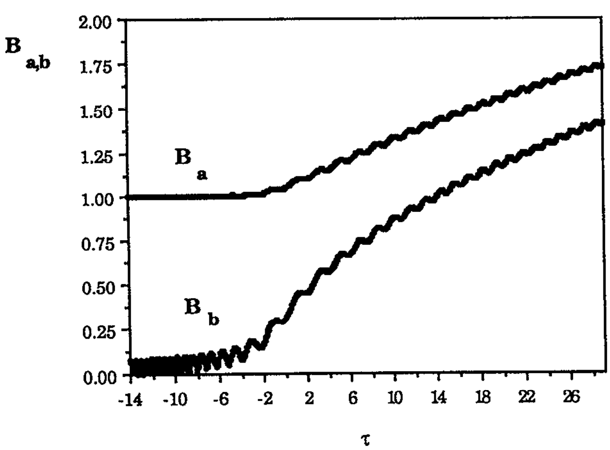

FIG. 8. The evolution of the magnitudes of the coupled modes $B_{a,b}$ in the unstable coupling example shown in Fig. 7. The amplification continues indefinitely due to the autoresonance.

VI. CONCLUSIONS

(i) This work comprises an investigation of the effect of nonlinearities on the conventional pairwise linear mode conversion.

(ii) We have generalized the congruent reduction technique for applications to weakly nonlinear situations, and have applied the technique to reducing the weakly nonlinear conservative coupled mode equations for a general geometry.

(iii) The solutions of the coupled mode equations were studied analytically and numerically in the one-dimensional case. It was shown that the effect of the nonlinearities on the mode conversion can be significant if the system enters the spatial autoresonance (SAR) regime. The conditions for the SAR are (a) the launching of mostly a single mode at the boundary of the region containing the crossing point [inequality (46)]; (b) the smallness of the ratio μ/β between the normalized coupling and nonlinearity parameters [condition (36), independent of the degree of the inhomogeneity]; and (c) a sufficiently large value of the product $\mu\beta$ [inequality (61)].

(iv) The details of the autoresonance phase of the interaction depend on whether the action density signs of the coupled modes are the same or different. In the former case (the stable coupling), the action flux from the initially excited mode is efficiently transferred to the second mode and then the interaction stops, while in the latter case (the unstable coupling), both modes are continuously amplified in space until the weak nonlinearity assumption is violated and a different approach to the problem is needed.

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