PHYSICAL REVIEW E VOLUME 48, NUMBER 4 OCTOBER 1993

Autoresonant interaction of three nonlinear adiabatic oscillators

Shalom Yariv and Lazar Friedland
Center for Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel
(Received 8 March 1993)

The autoresonance phenomenon in a system of three weakly interacting nonlinear oscillators with slowly varying parameters is studied. When excited, the triple autoresonant interaction yields an efficient and continuous energy exchange between the oscillators while the nonlinear resonance condition is preserved, despite the variation of the parameters of the system. It is shown that the autoresonance is stable with respect to the addition of sufficiently small friction and fluctuations in the system. The amount of the allowed noise is estimated and the theory is illustrated by numerical examples.

PACS number(s): 03.20.+i, 03.40.Kf, 52.35.Mw, 05.40.+j

I. INTRODUCTION

In contrast to the harmonic oscillator, the frequencies of nonlinear oscillators depend on their energy, or action. Because of this dependence, it was noticed long ago [1,2] that one can *resonantly* transfer a nonlinear oscillator from any of its allowed energy states to another state by applying an external alternating perturbing force with a *slowly varying* frequency. If, initially, the system is in resonance, then later in time the frequency of the nonlinear oscillator will follow the frequency of the driving perturbation so that the resonance condition is preserved on average. The energy of the nonlinear oscillator changes accordingly. This effect was called the *phase stability* in earlier studies [2] and the dynamic *autoresonance* (DAR) in more recent works [3,4].

In studying systems of weakly interacting nonlinear oscillators with constant parameters the most important are the pairwise resonant interactions between the oscillators in the system. When such resonances are isolated one can understand the system's dynamics by using the *single resonance* approximation [5,6], since the nonresonant contributions are usually averaged out. If the parameters of the interacting oscillators vary adiabatically, then one again finds the DAR type interactions between *pairs* of oscillators in the system which adjust their energy states so that the nonlinear resonance condition is preserved despite the time variation of the parameters. An important case of this type is the *spatial autoresonance* in mode conversion [7], which is the spatial analog of the DAR for multicomponent weakly nonlinear waves in nonuniform medium.

If pairwise resonance conditions in a system of weakly interacting oscillators are not satisfied, then resonances between more than two oscillators in the system may become important. For instance, three-wave resonant interactions in weakly nonlinear systems supporting multicomponent waves comprise the lowest order (in terms of the wave amplitudes) nonlinear resonant effect and are therefore very important. The generalization of the DAR idea for applications to such systems was suggested recently [8]. In the present work we also consider the problem of the triple resonance by using a model of three weakly interacting oscillators with slowly varying parameters, but remove the weak nonlinearity assumption of previous studies. We shall show that if the system is resonant initially, then, under certain conditions, the problem reduces to that of the nonlinear resonance in a system of one degree of freedom and yields the autoresonant solution for which the triple resonance is preserved despite the variation of the parameters of the system. Finally, we shall study the effect of a weak dissipation (friction) and fluctuations (collisions) on the stability of the triple DAR.

II. THE MODEL EQUATIONS

A classical system of three weakly interacting oscillators exhibiting the triple resonance phenomenon is described by the Hamiltonian

$$H(\mathbf{x},\mathbf{p},t)=\sum_{i=1}^{3}\left[\frac{p_j^2}{2}+\frac{\Omega_j^2x_j^2}{2}\right]+\varepsilon x_1x_2x_3\ , \tag{1}$$

where ε is a small parameter. In the past this model served as a prototype for studying resonant three-wave interactions, where the term $\varepsilon x_1x_2x_3$ represented the lowest-order nonlinear effect allowing the resonances of the type $\Omega_1\pm\Omega_2\pm\Omega_3=0$. Important generalizations of (1) included the possibility of adiabatic variations of the parameters of the system (such as frequencies Ω_j) and the addition of weakly nonlinear frequency shifts [9–11]. In this work we further generalize the theory by removing the weak nonlinearity assumption on the oscillators, but still allow adiabatic variations of system's parameters and preserve the interaction term as in (1). Thus, our Hamiltonian is

$$H(\mathbf{x},\mathbf{p},t)=\sum_{j=1}^{3}\left[\frac{p_j^2}{2}+V_j(x_j,\lambda_j(t))\right]+\varepsilon x_1x_2x_3\ , \tag{2}$$

where $\lambda_j(t)$ represents a set of slowly varying parameters characterizing the jth oscillator.

At this point it is convenient to introduce the action-angle variables $(\mathbf{I}_j,\varphi_j)$, $j=1,2,3$ of the unperturbed oscillators instead of (p_j,x_j). Then (2) yields the following equations of motion:

$$\dot{\mathbf{I}}_\alpha = -\varepsilon \frac{\partial x_\alpha}{\partial \varphi_\alpha} x_\beta x_\gamma \ , \tag{3}$$

$$\dot{\varphi}_\alpha = \Omega_\alpha + \varepsilon \frac{\partial x_\alpha}{\partial \mathbf{I}_\alpha} x_\beta x_\gamma \ , \tag{4}$$

with $\alpha = 1,2,3$, $\beta \neq \gamma \neq \alpha$, and Ω_α representing the nonlinear frequencies of the unperturbed oscillators. Next, we can expand x_α in Fourier series in terms of the angle variables:

$$x_\alpha = \sum_{k=-\infty}^{\infty} a_\alpha^k(\mathbf{I}_\alpha, \lambda_\alpha) e^{ik\varphi_\alpha}, \quad a_\alpha^{-k} = a_\alpha^{+k*} \ . \tag{5}$$

If (5) is substituted in (4), and only the resonant contribution in the interaction term $k\Omega_1 + m\Omega_2 + n\Omega_3$ is kept, we obtain the following system of equations:

$$\dot{\mathbf{I}}_\alpha = 2\varepsilon k_\alpha \operatorname{Im}\{F e^{i\sigma}\} \ , \tag{6}$$

$$\dot{\sigma} = (k\Omega_1 + m\Omega_2 + n\Omega_3) + 2\varepsilon \operatorname{Re}\{G e^{i\sigma}\} \ , \tag{7}$$

where k_α is either k, m, or n depending on whether α is 1, 2, or 3, $\sigma = k\varphi_1 + m\varphi_2 + n\varphi_3$, and

$$F = a_1^k a_2^m a_3^n \ , \tag{8}$$

$$G = F \left[\frac{k}{a_1^k} \frac{\partial a_1^k}{\partial \mathbf{I}_1} + \frac{m}{a_2^m} \frac{\partial a_2^m}{\partial \mathbf{I}_2} + \frac{n}{a_3^n} \frac{\partial a_3^n}{\partial \mathbf{I}_3} \right] . \tag{9}$$

Equation (6) yields the Manley-Rowe conditions:

$$\mathbf{I}_\alpha / k_\alpha - \mathbf{I}_\beta / k_\beta = C_{\alpha\beta} = \text{const}, \quad \alpha \neq \beta \ , \tag{10}$$

which allow one to express any pair of the amplitudes in terms of the third amplitude, so we have just two independent equations, say, for $\mathbf{I}_1$ and σ. As a final preliminary step, we now introduce the real amplitudes and complex phases of a_α^k, i.e., $a_\alpha^k \equiv B_\alpha^k \exp(i\psi_\alpha^k)$, and define the phase shift $\phi = k\psi_1^k + m\psi_2^m + n\psi_3^n + \sigma$. Then (5) and (6) can be rewritten as

$$\dot{\mathbf{I}}_\alpha = 2\varepsilon k_\alpha \bar{F} \sin\phi \ , \tag{11}$$

$$\dot{\phi} = \omega + 2\varepsilon \bar{G} \cos\phi \ , \tag{12}$$

where $\bar{F}$ and $\bar{G}$ are given by (8) and (9) with a_α^k replaced by B_α^k and $\omega \equiv k\Omega_1 + m\Omega_2 + n\Omega_3$.

In the case of constant parameters in the system (where ω is independent of time), Eqs. (11) and (12) are similar to the well-known equations characteristic of the classical nonlinear resonance [6]. In this case and when ε satisfies

$$\varepsilon \ll \frac{d\omega}{d\mathbf{I}_\alpha} \frac{1}{\omega} \ , \tag{13}$$

this system yields, in the vicinity of the resonance $\omega \approx 0$, a solution in which both $\mathbf{I}_\alpha$ and ϕ exhibit an oscillatory behavior with a characteristic frequency [5,6]

$$\nu \propto \left[2\varepsilon k\alpha \bar{F} \frac{d\omega}{d\mathbf{I}_\alpha} \right]^{1/2} \ , \tag{14}$$

while the amplitudes $\delta\mathbf{I}_\alpha$ and $\delta\phi$ of these oscillations [and therefore also the amplitudes of $\delta\omega = \delta\mathbf{I}_\alpha(\partial\omega/\partial\mathbf{I}_\alpha)$] are small and of order $\varepsilon^{1/2}$.

If the parameters of the oscillators vary adiabatically, i.e., ω is a function of time, but

$$\frac{d\omega}{dt} \ll \nu^2 \ , \tag{15}$$

then the system exhibits the DAR phenomenon [3,4,7,8], where $\mathbf{I}_\alpha$ and ϕ_α again oscillate, however, $\langle \omega(\mathbf{I}_\alpha, t) \rangle_{\text{av}} = 0$ despite the variation of parameters with time. In other words, $\langle \mathbf{I}_\alpha \rangle_{\text{av}}$ adjusts itself automatically in time to preserve the nonlinear resonance condition.

An important problem associated with the DAR phenomenon is the passage through the resonance. Generally, if at the initial interaction time, ω is far from the resonance by more than its characteristic width (which scales as $\sqrt{\varepsilon}$), the system cannot enter the autoresonance by varying its parameters adiabatically, since the trapping into the resonance requires the crossing of the separatrix between the trapped and untrapped trajectories in the $(\mathbf{I}_\alpha, \phi)$ phase plane. Nevertheless, an important exception exists in our system, when, initially, one of the oscillators, say, oscillator 1, is not excited significantly. In this case, if the fundamental harmonic $(k = \pm 1)$ of this oscillator is of interest, the trapping is guaranteed in spite of the adiabaticity of the system. This effect was noticed in Ref. [7] and is due to the fact that for a weakly excited oscillator, one usually has $a_1^{\pm 1} \sim \sqrt{\mathbf{I}_1}$ and therefore $\bar{G}$ in (12) is singular. A simple analysis then shows that this singularity leads to the above-mentioned trapping into the resonance [7].

At this stage, we shall proceed to an example. Consider the case of two linear oscillators, denoted by subscripts 1 and 2, and one nonlinear oscillator, denoted by 3. We choose the frequencies of the linear oscillators to be $\Omega_{1,2}(t)$, while the third oscillator corresponds to a particle trapped in a stationary square well potential in the region $|x_3| < l/2$. For the linear oscillators with the equilibrium points at $x_{1,2} = 0$ we can write

$$\mathbf{I}_{1,2} = \frac{\Omega_{1,2}}{2} (x_{1,2}^{\max})^2 \tag{16}$$

and

$$x_{1,2} = \left[\frac{2\mathbf{I}_{1,2}}{\Omega_{1,2}} \right]^{1/2} \cos\varphi_{1,2} \ , \tag{17}$$

where $x_{1,2}^{\max}$ are the amplitudes of the oscillations. Therefore,

$$a_{1,2}^k = \begin{cases} \sqrt{\mathbf{I}_{1,2}/2\Omega_{1,2}}, & \text{for } k = \pm 1 \\ 0, & \text{otherwise} \ . \end{cases} \tag{18}$$

On the other hand, the nonlinear oscillator satisfies

$$\mathbf{I}_3 = l|u|/\pi \ , \tag{19}$$

where u is the velocity of the trapped particle. Then the nonlinear frequency is given by

$$\Omega_3 = \frac{\pi^2 \mathbf{I}_3}{l^2} \tag{20}$$

and

$$x_3=\frac{l}{2}\sum_{m=1}^{\infty}\frac{\cos(m\varphi_3)}{\pi^2 m^2} . \tag{21}$$

Hence

$$a_3^n=\begin{cases}0 \text{ for } n=0\\ l/(4\pi^2 n^2), \text{ otherwise} .\end{cases} \tag{22}$$

Now, consider the resonance $\Omega_1-\Omega_2-\Omega_3\approx 0$ ($k=1$, $m=-1$, $n=-1$). The Manley-Rowe relations [Eq. (10)] in this case are

$$\mathbf{I}_3+\mathbf{I}_1=C_{31} ,\qquad \mathbf{I}_3-\mathbf{I}_2=C_{32} , \tag{23}$$

and our system is fully described by the following set of equations:

$$\dot{\mathbf{I}}_3=-\frac{\varepsilon l}{4\pi^2\sqrt{\Omega_1\Omega_2}}[(C_{31}-\mathbf{I}_3)(C_{32}+\mathbf{I}_3)]^{1/2}\sin\phi , \tag{24}$$

$$\dot{\phi}=\left[\Omega_1-\Omega_2-\frac{\pi^2}{l^2}\mathbf{I}_3\right]+\frac{\varepsilon l}{8\pi^2\sqrt{\Omega_1\Omega_2}}\times\left[\frac{(C_{31}-\mathbf{I}_3)^{1/2}}{(C_{32}+\mathbf{I}_3)}+\frac{(C_{32}+\mathbf{I}_3)^{1/2}}{(C_{31}-\mathbf{I}_3)}\right]\cos\phi . \tag{25}$$

Assume, first, that $\Omega_{1,2}$=const. Then the system has stable and unstable fixed points, given by

$$\cos\phi=\pm 1 \tag{26}$$

and

$$\left[\Omega_1-\Omega_2-\frac{\pi^2}{l^2}\mathbf{I}_3\right]+\frac{\varepsilon l}{8\pi^2\sqrt{\Omega_1\Omega_2}}\times\left[\frac{(C_{31}-\mathbf{I}_3)^{1/2}}{(C_{32}+\mathbf{I}_3)}+\frac{(C_{32}+\mathbf{I}_3)^{1/2}}{(C_{31}-\mathbf{I}_3)}\right]=0 , \tag{27}$$

and therefore the steady-state solutions are

$$\phi_*=0,\pi , \tag{28}$$

$$\mathbf{I}_{3*}=(l^2/\pi^2)(\Omega_1-\Omega_2)+o(\varepsilon) \tag{29}$$

and a simple linear analysis then shows that the stable phase corresponds to $\phi_*=\pi$.

Next we allow the adiabatic variation of the parameters (in our example the frequencies of the linear oscillators). We observe, that the right-handside of Eq. (25) has two terms. The first term $\Delta\Omega=(\Omega_1-\Omega_2-\Omega_3)$ is small near the resonance. The second term may be large if either $\mathbf{I}_1$ or $\mathbf{I}_2$ is small. As mentioned earlier, in such a case, this singularity causes the trapping into the resonance [7]. Due to the variation of $\Omega_{1,2}$ with time, one approaches the point where $\Delta\Omega\to 0$. After the trapping, the $O(\varepsilon)$ term in (25) can be neglected. Now we seek solutions for Eqs. (23) and (24) in the form

$$\mathbf{I}_3=\mathbf{I}_{3*}(t)+\delta\mathbf{I} , \tag{30}$$

where $\mathbf{I}_{3*}(t)$ is given by (29) and $\delta\mathbf{I}$ is of $O(\varepsilon^{1/2})$ and oscillates. Then, to order $O(\varepsilon^{1/2})$, Eqs. (24) and (25) give

$$\delta\dot{\mathbf{I}}=-\frac{\varepsilon l}{4\pi^2}f(\mathbf{I}_{3*})\sin\phi-\left[\frac{l^2}{\pi^2}\right]\left[\dot{\Omega}_1-\dot{\Omega}_2\right] , \tag{31}$$

$$\dot{\phi}=-\frac{\pi^2}{l^2}\delta\mathbf{I} , \tag{32}$$

where

$$f(\mathbf{I}_{3*})=\frac{[(C_{31}-\mathbf{I}_{3*})(C_{32}+\mathbf{I}_{3*})]^{1/2}}{\sqrt{\Omega_1\Omega_2}} . \tag{33}$$

The stable quasi-steady-state solution of (31) and (32) is

$$\phi_*=\pi-\arcsin\left[\frac{4l(\dot{\Omega}_1-\dot{\Omega}_2)}{\varepsilon f(\mathbf{I}_{3*})}\right] , \tag{34}$$

$$\delta\mathbf{I}=0 , \tag{35}$$

and exists only if

$$\left|\frac{4l(\dot{\Omega}_1-\dot{\Omega}_2)}{\varepsilon f(\mathbf{I}_{3*})}\right|<1 . \tag{36}$$

Equations (31) and (32) are Hamilton-type, and are similar to those characteristic of the classical nonlinear resonance in which the parameters vary slowly in time [in this case $\mathbf{I}_{3*}=\mathbf{I}_{3*}(t)$ as follows from (29)]. For instance, the differentiation of (32) yields the adiabatic nonlinear pendulum-type equation

$$\ddot{\phi}=\frac{\varepsilon f(\mathbf{I}_{3*})}{4l}\sin\phi-(\dot{\Omega}_1-\dot{\Omega}_2) . \tag{37}$$

The characteristic frequency of this pendulum $\nu=\sqrt{\varepsilon f/4l^2}$ then scales as $\sim\sqrt{\varepsilon}$, and the resonance width i.e., the maximum value of $|\delta\mathbf{I}|$ in the trapped region of the phase space, can then be estimated as

$$\Delta\mathbf{I}=\left[\frac{l^3 f(\mathbf{I}_{3*})}{2\pi^4}\varepsilon\right]^{1/2} . \tag{38}$$

III. DISSIPATION

Now we depart from the Hamiltonian description by introducing the simplest dissipation term in Eq. (24) for the action, while leaving the equation for the phase ϕ unchanged:

$$\dot{\mathbf{I}}_3=-\frac{\varepsilon l}{4\pi^2\sqrt{\Omega_1\Omega_2}}[(C_{31}-\mathbf{I}_3)(C_{32}+\mathbf{I}_3)]^{1/2}\sin\phi-\eta\mathbf{I}_3 , \tag{39}$$

$$\dot{\phi}=\left[\Omega_1-\Omega_2-\frac{\pi^2}{l^2}\mathbf{I}_3\right] . \tag{40}$$

If we repeat the same procedure that was used to obtain (31) and (32), we get

$$\delta\dot{\mathbf{I}}=-\frac{\varepsilon l}{4\pi^2}f(\mathbf{I}_3)\sin\phi-\left[\frac{l^2}{\pi^2}\right](\dot{\Omega}_1-\dot{\Omega}_2)-\eta\delta\mathbf{I} , \tag{41}$$

$$\dot{\phi}=-\frac{\pi^2}{l^2}\delta \mathbf{I} . \tag{42}$$

We find the DAR effect $\Omega_1-\Omega_2-\langle\Omega_3\rangle=0$, provided [compare to (36)]

$$\left|\frac{4l(\dot{\Omega}_1-\Omega_2)+\eta(\Omega_1+\Omega_2)}{\varepsilon f(\mathbf{I}_{3*})}\right|<1 , \tag{43}$$

while the pendulum equation analogous to (37) is

$$\ddot{\phi}=\frac{\varepsilon f(\mathbf{I}_{3*})}{4l}\sin\phi-(\dot{\Omega}_1-\dot{\Omega}_2)-\frac{\pi^2}{l^2}\eta\dot{\phi} . \tag{44}$$

Thus, we conclude that the addition of a sufficiently small dissipation still allows a stable DAR effect with a nonlinear resonance preserved for an extended period of time despite both the variation of the parameters and the dissipation. In the special case when the parameters of the system are constant, the weak dissipation does not destroy the resonance condition $\Omega_1-\Omega_2-\langle\Omega_3\rangle=0$ because the energy dissipated from the nonlinear oscillator is replenished at the expense of the energy of the linear oscillators, as long as Eq. (43) is satisfied, i.e., as long as the energy of both linear oscillators is sufficiently large. We demonstrate this effect in Fig. 1 showing the evolution of $\mathbf{I}_{1,2,3}$ for $\Omega_1=7$, $\Omega_2=5$, and the initial value of $\Omega_3=2$. The values $\eta=10^{-3}$, $\varepsilon=10^{-2}$, and the initial conditions $x_1=0.0$, $x_2=1.0$, $x_3=0.2$, $\dot{x}_1=-2.5$, $\dot{x}_2=0.0$, $\dot{x}_3=1.273$ were used in these calculations. We see in the figure that the action $\mathbf{I}_3$ of the nonlinear oscillator remains constant on average despite the dissipation, as required by the resonance condition for constant $\Omega_{1,2}$. $\mathbf{I}_{1,2}$ in contrast, vary in time as their energy is transferred to the nonlinear oscillator. The DAR interaction disappears when one of the linear oscillators almost completely loses its energy, violating (43). In contrast, Fig. 2 shows the evolution of $\mathbf{I}_{1,2,3}$ in the case when $\Omega_1=7.0+5\times10^{-4}t, \Omega_2=5.0, \Omega_3=2.670$ initially (the system is out of resonance), $\varepsilon=0.01$, $\eta=3.5\times10^{-5}$, and initially $x_1=2.0$, $x_2=0.01$, $x_3=0.2$, $\dot{x}_1=-1.0$, $\dot{x}_2=0.0$, $\dot{x}_3=1.7$. The trapping into the resonance is seen in the figure. After the trapping, the linear time evolution of $\langle\mathbf{I}_3\rangle$ is evident and illustrates the adjustment of $\langle\Omega_3\rangle$ to the linear time dependence of Ω_1. The detrapping occurs in this case because of the breakdown of condition (43).

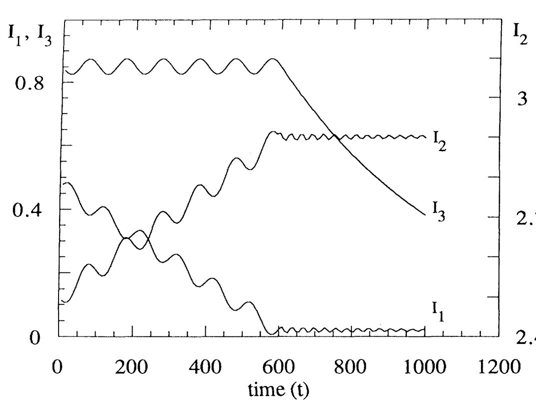

FIG. 1. The dependence of the actions of the three interacting oscillators on time. The angular frequencies of the two linear oscillators 1 and 2 are $\Omega_1=7.0$ and $\Omega_2=5.0$, and remain constant. The friction coefficient $\eta=0.001$, $\varepsilon=0.01$, and the initial conditions are $x_1=0.0$, $x_2=1.0$, $x_3=0.2$, $\dot{x}_1=-2.5$, $\dot{x}_2=0.0$, $\dot{x}_3=1.273$.

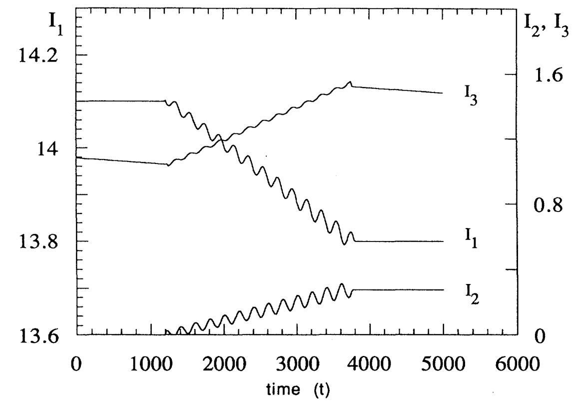

FIG. 2. The dependence of the actions of the three interacting oscillators on time when started outside the resonance. The angular frequencies of the two linear oscillators 1 and 2 are $\Omega_1=7.0+0.0005t$ and $\Omega_2=5.0$. The friction coefficient $\eta=0.000\,035$, $\varepsilon=0.01$, and the initial conditions are $x_1=2.0$, $x_2=0.01$, $x_3=0.2$, $\dot{x}_1=-1.0$, $\dot{x}_2=0.0$, $\dot{x}_3=1.7$. The three actions are shown.

IV. FLUCTUATIONS

In order to study the effect of fluctuations on the autoresonance interaction we choose a model in which our nonlinear oscillator collides with a gas of light particles. We use the same model for the three oscillators, so that, as before, the action $\mathbf{I}_3$ of the nonlinear oscillator is given by Eq. (19) where we choose $l=1$. We assume that if the nonlinear oscillator has a velocity u, then after a collision it becomes

$$\bar{u}=u+\delta(v-u) , \tag{45}$$

where $\delta \ll 1$ is twice the ratio between the mass of the gas particle and the mass of the oscillator, and v is the velocity of the gas particle. We also assume that the velocities of the gas particles are characterized by some symmetric (in the one-dimensional velocity space) distribution function with $\langle v^2\rangle=\beta^2$ and that $\beta^2 \gg u^2$.

Because of the collisions, the velocity of our nonlinear oscillator will both drift and diffuse in velocity space. The drift is due to the damping rate which is

$$\nu=\frac{\langle\bar{u}-u\rangle_{\rm av}}{\tau u}=-\frac{\delta}{\tau} , \tag{46}$$

where the averaging is over the distribution of the gas particles and τ is the average time between the collisions. The effective diffusion coefficient in velocity space of the nonlinear oscillator is

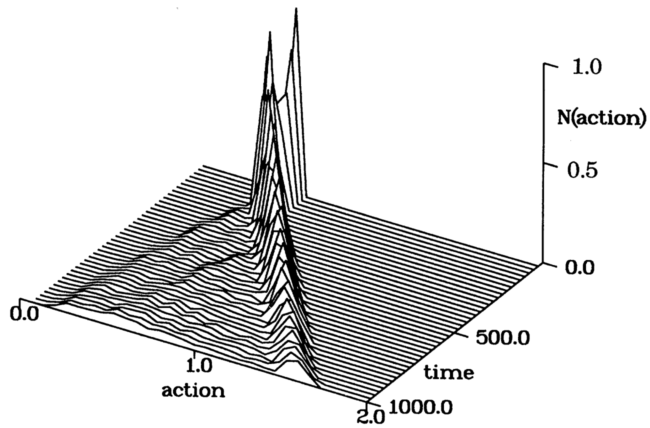

FIG. 3. The time evolution of the distribution function of actions of the nonlinear oscillator. The parameters and the initial values are $\varepsilon=0.05$, $\tau=10.0$, $\delta=0.004$, $\beta=1.0$, $l=2$, $\dot{x}_1=-1.0$, $\dot{x}_2=0.001$, $\dot{x}_3=1.0$, $x_1=2.0$, $x_2=0.01$, $\Omega_1=6.0+0.0003t$, $\Omega_2=5.0$, so that $t^*\approx 200$, and the competition between the autoresonance and diffusion can be observed.

$$D_u=\frac{\langle(\bar{u}-u)^2\rangle}{\tau}=\frac{1}{\tau}\left[\frac{\delta}{\beta}\right]^2 \tag{47}$$

and therefore the diffusion coefficient in action space is

$$D_{I_3}=\frac{1}{\tau}\left[\frac{l\delta}{\pi\beta}\right]^2 . \tag{48}$$

We now proceed to a numerical example. Figure 3 shows the evolution of the distribution of the velocities of the nonlinear oscillator at different times in the case $\varepsilon=0.05$, $\tau=10.0$, $\delta=0.002$, $\beta=1.0$, $l=2$, $\dot{x}_1=-1.0$, $\dot{x}_2=0.001$, $\dot{x}_3=1.0$, $x_1=2.0$, $x_2=0.01$, $\Omega_1=6.0+0.0003t$, and $\Omega_2=5.0$. We modeled the gas-particle velocity distribution function by $(1/\beta)\exp(-|v|/\beta)$ and used the conventional Monte Carlo technique to simulate the collisions. One can see in the figure that, initially, the effect of collisions is small and the nonlinear oscillator is effectively trapped into the triple resonance. Nevertheless, later, the distribution broadens due to the collisions and a part of the distribution escapes the resonance for times greater than t^* (which equals 200 in our example). This time can be estimated from

$$\sqrt{D_{I_3}t^*}=\Delta I , \tag{49}$$

where ΔI is the width of the resonance [see Eq. (38)]. Thus

$$t^*=\varepsilon\frac{l^3 f}{2\pi^2}\frac{\tau\beta^2}{\delta^2} . \tag{50}$$

V. CONCLUSIONS

We have studied the dynamics of three weakly interacting oscillators with a strong nonlinearity and slowly varying parameters. The single-resonance approximation was used in the analysis of the triple DAR in the system. It was demonstrated that the trapping into the resonance during the passage through the resonance takes place if one of the oscillators is only weakly excited initially. It was also shown analytically and demonstrated by numerical examples that the autoresonance is stable with respect to the addition of sufficiently weak damping. Finally, we have studied the effects of fluctuations on the DAR. These fluctuations were modeled via collisions with light gas atoms and simulated numerically by using the Monte Carlo method. It was shown that the autoresonance is preserved in the presence of collisions until the system escapes the resonance because of the action space diffusion that is due to collisions. The characteristic escape time was estimated and verified in simulations.

[1] D. Bohm and L. L. Foldy, Phys. Rev. **70**, 649 (1947).
[2] J. Rosenblatt, *Particle Acceleration* (Mathuen, London, 1968).
[3] B. Meerson and L. Friedland, Phys. Rev. A **41**, 5233 (1990).
[4] B. Meerson and S. Yariv, Phys. Rev. A **44**, 3570 (1991).
[5] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).
[6] B. V. Chirikov, Phys. Rep. **52**, 263 (1978).
[7] L. Friedland, Phys. Fluids B **4**, 3199 (1992).
[8] L. Friedland, Phys. Rev. Lett. **69**, 1749 (1992).
[9] M. N. Rosenbluth, Phys. Rev. Lett. **29**, 565 (1972).
[10] T. A. Davydova and V. N. Oraevskii, Zh. Eksp. Teor. Fiz. **66**, 1613 (1974) [Sov. Phys. JETP **39**, 791 (1974)].
[11] V. N. Oraevskii, in *The Handbook of Plasma Physics*, edited by A. Galeev and R. Sudan (North-Holland, Amsterdam, 1983), Vol. 2, p. 37.