

Spatial autoresonance cyclotron accelerator

Lazar Friedland^{a)}

Laboratoire de Physique des Milieux Ionises, Laboratoire du C.N.R.S., Ecole Polytechnique, 91128 Palaiseau, France

(Received 18 August 1993; accepted 26 October 1993)

A mechanism of electron acceleration scheme utilizing the spatial autoresonance phenomenon in combined axial guide magnetic field and a traveling electromagnetic wave with adiabatically varying parameters is analyzed. The acceleration is achieved due to the self-tendency of the particles to stay in the relativistic gyroresonance with the wave, despite the variation of systems parameters. As compared to other traveling wave cyclotron acceleration schemes, the spatial autoresonance accelerator does not require a precise initial frequency matching and a rigid tuning of the guide magnetic field for sustaining the resonance. The method is illustrated in the case of a microwave driver with a tapered axisymmetric waveguide geometry.

I. INTRODUCTION

The use of the relativistic gyroresonance for acceleration of charged particles was suggested in early studies^{1,2} of the interaction between an electron in a uniform guide magnetic field, and a transverse electromagnetic wave. The idea of using this configuration as an accelerator followed the discovery of the "synchronous" interaction regime,³ when the particle, starting in the gyroresonance, i.e., satisfying, initially,

$$\omega - kv_z - \Omega_c/\gamma = 0 \quad (1)$$

[here ω and k are the frequency and wave vector of the wave (propagating in the z direction), v_z and (later) v_\perp are the longitudinal and transverse electron velocity components with respect to the direction of the guide magnetic field $\mathbf{B} = B_0 \hat{e}_z$, $\gamma^{-2} = 1 - (v_\perp/c)^2 - (v_z/c)^2$ is the usual relativistic factor, and $\Omega_c = eB_0/m_e c$ is the cyclotron frequency] stays in resonance *indefinitely*, despite the variation of both v_z and γ as the particle moves along the guide field. The "synchronous" regime requires (a) the satisfaction of the *exact* gyroresonance condition initially, and (b) the use of the *luminous* ($n \equiv ck/\omega = 1$), circularly polarized electromagnetic wave. For these synchronous conditions the particle is accelerated indefinitely, with γ scaling with distance (for the electron initially at rest) as² $\gamma = [(3/\sqrt{2})\epsilon z]^2/3$, where $\epsilon = eE/m_e c^2$ is the normalized amplitude of the electric field of the wave. The scaling exponent of $2/3$ is smaller than that in linear accelerators where $\gamma = \epsilon z$ (assuming the same value for ϵ in both cases). Nevertheless, for final values of γ lower than 4.5, the gyroresonant accelerator is more efficient.²

Despite the advantages of the gyroresonant acceleration at lower energies, its main deficiency lay in the strict requirement of the satisfaction of the above-mentioned synchronous conditions. Any violation of these conditions, due to either the nonluminous realization of the electromagnetic wave or inaccuracies in the initial frequency matching in practical applications, led to the departure

from (1) for large z , thus setting a limit on the maximum achievable acceleration. Nevertheless, when the initial conditions on the electron beam are such that Eq. (1) is satisfied initially, and the *only* problem is the nonluminosity of the wave, one can still overcome the dephasing problem. This goal can be achieved by tapering the guide magnetic field with z , so that the gyroresonance condition is *imposed* continuously. This idea was put forward in early studies of this scheme² and exploited later in applications with optical (laser) fields.^{4,5} In practice, in this case, one must solve the equations of motion for the electrons subject to Eq. (1), as an additional constraint, thus defining the necessary spatial tapering of the guide field. The scheme is obviously dependent on the initial conditions, and each set of such conditions (for a given electromagnetic field configuration) yields a different tapering. Thus, we shall refer to this scheme as the *rigid tapering* accelerator in the following. This rigid tapering approach attracted renewed attention in recent years, and was a subject of investigation in a number of theoretical studies.⁴⁻⁶ Furthermore, the proof-of-the-principle, rigid tapering-type experiment was performed recently.⁷ It showed that, as expected, the main limitation of the rigid tapering scheme was its sensitivity to the uncertainties in initial conditions and other unavoidable inaccuracies.

Despite the aforementioned activity associated with the rigid tapering acceleration method, there remained unnoticed another gyroresonant acceleration approach, exploiting a similar configuration of the guide and electromagnetic fields, but, which, in contrast, does not require the exact initial satisfaction of the gyroresonance condition and a strict tapering of the guide field. This scheme also utilizes variation of systems parameters (the wave vector in the case considered below), but the tapering is not rigid and must only be sufficiently slow. Such a non-rigid acceleration regime in the system can be viewed as the spatial analog of the *autoresonance* effect, suggested earlier as the basis for the gyroresonance cyclic accelerator (GYRAC).^{8,9} The GYRAC uses a time-dependent guide magnetic field and a standing transverse electromagnetic wave. If the time variation of the guide field is sufficiently slow, and one initially starts in the *vicinity* of the gyrores-

^{a)}On sabbatical from the Center of Plasma Physics, Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem, Israel.

onance, i.e., $(\omega - \Omega_c/\gamma)_{t=0} \approx 0$, then the electron slowly oscillates around the resonance, but, nevertheless, stays in the gyroresonance in *average* over the period of these oscillations.^{8,9} This result in the GYRAC does not depend on the actual rate of the variation of the guide field (as long as this variation is sufficiently slow), and yields a very simple scaling, $\langle \gamma(t) \rangle_{av} = \Omega_c(t)/\omega$, so that one can obtain a significant acceleration as the guide field increases in time. The autoresonance idea, used in Refs. 8 and 9, was an application of a more general *dynamic autoresonance* (DAR) phenomenon characteristic of resonantly perturbed nonlinear oscillators with slowly varying parameters. Other applications of the DAR were recently suggested in atomic physics,¹⁰ laser-induced generation of intense plasma waves¹¹ and nonlinear dynamics.^{12,13} Also, *spatial* analogs of the DAR were found in mode conversion¹⁴ and three-wave resonant interactions.¹⁵ In the present work we propose a *spatial* analog of the GYRAC autoresonance acceleration scheme, i.e., the *spatial autoresonance cyclotron accelerator* (or SACA).

Our presentation will be as follows. We shall develop the basic set of equations characterizing our accelerator configuration in Sec. II. The constant wave vector case will be studied in Sec. III forming the necessary basis for the spatial autoresonance theory presented in Sec. IV. In Sec. V, seeking the autoresonant acceleration for a wider range of initial conditions, we shall consider the problem of the trapping into the resonance. In Sec. VI we will deal with the scaling laws in our accelerator. Numerical examples, illustrating our theory in the case of a microwave driver with a tapered axisymmetric waveguide geometry, will be also presented in Sec. VI. Finally, we shall summarize our results in Sec. VII.

II. MODEL EQUATIONS

Consider the motion of an electron in combined *uniform* guide magnetic field $\mathbf{B} = B_0 \hat{e}_z$ and a circularly polarized electromagnetic wave of the form

$$\begin{aligned} \mathbf{E}(z,t) &= E(z) (\hat{e}_x \sin \psi + \hat{e}_y \cos \psi), \\ \mathbf{B}(z,t) &= [ck(z)/\omega] E(z) (\hat{e}_y \sin \psi - \hat{e}_x \cos \psi), \end{aligned} \quad (2)$$

where $\psi(z,t) \equiv \int k(z) dz - \omega t$, and we assume that the frequency ω is constant, but the wave vector $k(z)$ is a *slowly* varying function of position z . The wave vector $k(z)$ and the frequency ω are related via a local dispersion relation $D[k(z), \omega, z] = 0$ (which need not be specified at the moment), and the *slow* z dependence of the amplitude of the electric field is governed by the divergence of the electromagnetic flux. We shall neglect the effect of the electron beam on the driving electromagnetic wave, limiting the discussion to the case of a small overall accelerator efficiency (low beam current). Nevertheless, because of the stability of the spatial autoresonance acceleration mechanism with respect to *slow* variations of the parameters of the system (see below), SACA may be tolerant to self-consistent variation of the driving wave in the high current operation mode. The investigation of this effect is an important goal for future studies.

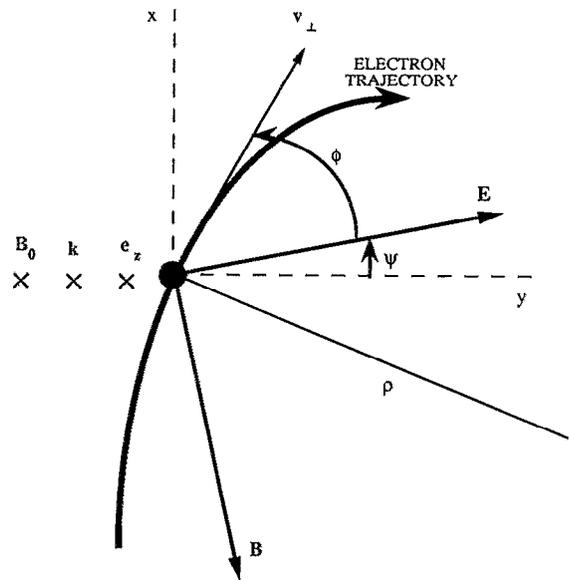


FIG. 1. The transverse projection of the electron trajectory.

Now we express the electron velocity as $\mathbf{v} = \mathbf{v}_\perp + \hat{e}_z v_z$, and write the centrifugal acceleration in our geometry (see Fig. 1):

$$-\frac{\gamma m_e v_\perp^2}{\rho} = eE \sin \phi - \left(\frac{ev_z B}{c} \right) \sin \phi - \frac{ev_\perp B_0}{c}. \quad (3)$$

Here ρ is the radius of the curvature of the trajectory in the transverse plane and ϕ is the angle between the directions of \mathbf{v}_\perp and \mathbf{E} . We also write the axial momentum equation,

$$\frac{d(\gamma v_z)}{dt} = - \left(\frac{ev_\perp B}{m_e c} \right) \cos \phi, \quad (4)$$

and the energy balance equation,

$$m_e c^2 \frac{d\gamma}{dt} = -eE v_\perp \cos \phi. \quad (5)$$

Equation (3) can be used in writing an expression for the "azimuthal" velocity:

$$\Omega \equiv \frac{-v_\perp}{\rho} = \left(\frac{eE}{m_e \gamma v_\perp} \right) \left(1 - \frac{v_z k}{\omega} \right) \sin \phi - \frac{\Omega_c}{\gamma}, \quad (6)$$

which, in turn, yields the rate of variation of ϕ along the trajectory,

$$\begin{aligned} \frac{d\phi}{dt} &= \Omega - \frac{d\psi}{dt} \\ &= \omega - kv_z - \frac{\Omega_c}{\gamma} + \left(\frac{eE}{m_e \gamma v_\perp} \right) \left(1 - \frac{v_z k}{\omega} \right) \sin \phi. \end{aligned} \quad (7)$$

Finally, we introduce the following rescaled and normalized variables and parameters:

$$\Phi = \frac{-\pi}{2} - \phi, \quad \mathbf{u} = \frac{\mathbf{v}}{c}, \quad \epsilon = \frac{eE}{m_e c^2}, \quad \omega_0 = \frac{\omega}{c}, \quad \Omega_0 = \frac{\Omega_c}{c}. \quad (8)$$

In these notations [and, since $(d/dt) = cu_z(d/dz)$], Eqs. (4), (5), and (7) can be rewritten as

$$\begin{aligned} \frac{d\gamma}{dz} &= \epsilon u_1 u_z^{-1} \sin \Phi, \\ \frac{d\phi}{dz} &= k(z) - u_z^{-1} \left(\omega_0 - \frac{\Omega_0}{\gamma} \right) \\ &\quad + \epsilon (\gamma u_1 u_z)^{-1} \left(1 - \frac{u_z}{u_p} \right) \cos \Phi, \\ \frac{d(\gamma u_z)}{dz} &= \epsilon u_1 (u_p u_z)^{-1} \sin \Phi, \end{aligned} \quad (9)$$

where $u_p \equiv \omega_0/k = n^{-1}$. Equations (9), in combination with the identity $\gamma = (1 - u_1^2 - u_z^2)^{-1/2}$, comprise a closed system of equations describing the *spatial* evolution of our system.

III. CONSTANT WAVE VECTOR CASE

In studying the solutions of (9), it is advantageous to proceed from the case of a constant wave vector k . We start by combining the first and the third equations in (9), yielding

$$\left(\frac{d}{dz} \right) [\gamma(1 - u_p u_z)] = -\gamma u_z \frac{du_p}{dz}. \quad (10)$$

Therefore, in the constant k case ($du_p/dz=0$), one has the following constant of motion:

$$L \equiv \gamma(1 - u_p u_z) = \text{const.} \quad (11)$$

On using (11), we can express u_z via γ ,

$$u_z = u_p^{-1} (1 - L/\gamma) \equiv U_z(\gamma). \quad (12)$$

On the other hand,

$$u_1^2 = \gamma^{-2} (\gamma^2 - 1) - U_z^2(\gamma) \equiv U_1^2(\gamma). \quad (13)$$

Therefore, for $k = \text{const}$, our system can be fully described by just two equations:

$$\begin{aligned} \frac{d\gamma}{dz} &= \epsilon f(\gamma) \sin \Phi, \\ \frac{d\Phi}{dz} &= k - \Omega(\gamma) + \epsilon g(\gamma) \cos \Phi, \end{aligned} \quad (14)$$

where $f \equiv U_1/U_z$, $g \equiv [\gamma U_1 U_z]^{-1} [1 - U_z/u_p]$, and $\Omega \equiv U_z^{-1} (\omega_0 - \Omega_0/\gamma)$.

A simple solution of Eqs. (7) in the vicinity of the gyroresonance [$k \approx \Omega(\gamma)$] exists when one can treat the terms with ϵ in (14) as a perturbation (the dimensionless smallness condition in such a case will be found later). In this case, we seek the solution for γ oscillating around an average value γ_0 given by the *exact* gyroresonance condition $k = \Omega(\gamma_0)$, and assume that the amplitude of these oscillations is small [formally of $O(\epsilon^{1/2})$]. Then, one can neglect the $O(\epsilon)$ term with $\cos \Phi$ in the second equation in (14), yielding $d\Phi/dz = k - \Omega(\gamma)$. By differentiating this equation and using the first equation in (14), we obtain

$$\frac{d^2\Phi}{dz^2} + \kappa_0^2 \sin \Phi = 0, \quad (15)$$

where, to the lowest order, $\kappa_0^2 \equiv \epsilon f(\gamma_0) [\partial\Omega/\partial\gamma]_{\gamma=\gamma_0}$. Thus, we have reduced the constant k case to that of a classical nonlinear "pendulum." If total initial "energy" of this "pendulum" is sufficiently low, then Φ oscillates in space with the characteristic "frequency" of $O(\kappa_0) \sim O(\epsilon^{1/2})$ around the equilibrium point $\Phi_0 = 0$. The first equation in (14) then shows that γ also oscillates around its equilibrium value γ_0 , and, as assumed, the amplitude $\Delta\gamma$ of these oscillations is of $O[\epsilon f(\gamma_0)/\kappa_0] \sim O(\epsilon^{1/2})$. By demanding, for consistency, that $\Delta\gamma/\gamma_0 \ll 1$, we obtain the desired dimensionless smallness condition on ϵ ,

$$\epsilon f(\gamma_0) \gamma_0^{-2} \left| \left(\frac{\partial\Omega}{\partial\gamma} \right)_{\gamma=\gamma_0} \right|^{-1} \ll 1. \quad (16)$$

Finally, the neglect of the term with $\cos \Phi$ in the second equation in (14) is justified, only if the amplitude $\Delta\Omega \approx (\partial\Omega/\partial\gamma)_{\gamma=\gamma_0} \Delta\gamma$ of the oscillations of Ω is large compared to $\epsilon g(\gamma_0)$, i.e.,

$$\epsilon [g(\gamma_0)]^2 \left| f(\gamma_0) \left(\frac{\partial\Omega}{\partial\gamma} \right)_{\gamma=\gamma_0} \right|^{-1} \ll 1. \quad (17)$$

In summarizing the constant k case, we have shown that if the inequalities (16) and (17) are satisfied, and if, initially, γ is in the vicinity of its gyroresonant value γ_0 ($|\gamma - \gamma_0| \sim O[\epsilon f(\gamma_0)/\kappa_0]$, i.e., $d\Phi/dz$ is small), then γ performs simple nonlinear spatial oscillations, but, nevertheless, satisfies (1) indefinitely if averaged over the period of these oscillations. Finally, we observe, that the spatial oscillations of γ are translated, via Eqs. (12) and (13), into spatially oscillating solutions for $u_{z,1}$.

IV. SPATIAL AUTO-RESONANCE

Now we proceed to the case $k = k(z)$, but assume that the spatial variation of k is sufficiently slow that the general *local* characteristic features of the solution in the $k = \text{const}$ case are preserved. Then, if we again start in the vicinity of the gyroresonance, we expect the solutions of (9) to be of the form $\gamma = \gamma_0(z) + \delta\gamma$, $u_z = u_{z0}(z) + \delta u_z$ (and similarly for u_1), where, now, the equilibrium values slowly evolve in space, but, as before, the oscillating parts $\delta\gamma$, $\delta u_{z,1}$ are small and of $O(\epsilon^{1/2})$. By treating du_p/dz also as small [of $O(\epsilon)$, see below], we can neglect the oscillating part in the RHS of (10) and rewrite this equation as

$$\frac{dL}{dz} = -\gamma_0 u_{z0} \frac{du_p}{dz}. \quad (18)$$

Therefore, $L \equiv \gamma(1 - u_p u_z)$ is not conserved as in the constant k case, but is a *slowly* varying function of position. Nevertheless, we can again use (12) and (13) as defining $U_{z,1}$, but now these functions depend explicitly not only on γ , but also (slowly) on z via $L(z)$. By using $U_{z,1}(\gamma, z)$, in the first two equations in (9), we arrive at the system similar to (14), but with an additional slow dependence on z in the wave vector $k(z)$ and functions f , g and Ω :

$$\frac{d\gamma}{dz} = \epsilon f(\gamma, z) \sin \Phi, \quad (19)$$

$$\frac{d\Phi}{dz} = k(z) - \Omega(\gamma, z) + \epsilon g(\gamma, z) \cos \Phi.$$

The system of equations of this form was studied recently in applications to mode conversion¹⁴ and three-wave resonant interactions.¹⁵ The spatial autoresonance was found in those applications, and, now, we shall proceed with a similar analysis in our case.

The analysis of the solutions of (19) can proceed similarly to that in the constant k case. Again, we adopt the perturbative procedure and assume that the inequalities (16) and (17) are satisfied locally, and, therefore, one can treat the terms of $O(\epsilon)$ in (19) as small. Then we can neglect the term with $\cos \Phi$ in the second equation in (19), differentiate this equation with respect to z , substitute $d\gamma/dz$ from the first equation in (19), and, to lowest order in ϵ , arrive at

$$\frac{d^2\Phi}{dz^2} + \kappa_0^2(z) \sin \Phi = \tau(z). \quad (20)$$

Here $\tau(z) \equiv (\partial/\partial z)[k(z) - \Omega(\gamma_0, z)]$ and $\kappa_0(z)$ is given by the same formula as in the constant k case, but now also depends on z due to the slowly varying equilibrium functions $\gamma_0(z)$, $u_{x0}(z)$, $u_{\perp 0}(z)$. Equation (20) can be viewed as describing the angle variable of an *adiabatic* nonlinear pendulum under the action of the small effective normalized "torque" τ . If κ_0 and τ in (20) would be constant, the problem could be described by a "stationary" effective Hamiltonian, $H_{\text{eff}}(\Phi, d\Phi/dz) = \frac{1}{2}(d\Phi/dz)^2 - \kappa_0^2 \cos \Phi - \tau\Phi$. The solution of this problem is simple, i.e., when τ is small enough, the phase plane $(\Phi, d\Phi/dz)$ of the "pendulum" can be separated into the regions of trapped and untrapped trajectories. The trapped region exists only if

$$|\tau| < \kappa_0^2, \quad (21)$$

since, otherwise, the effective potential $V_{\text{eff}} = -\kappa_0^2 \cos \Phi - \tau\Phi$ of the problem is a monotonically *decreasing* function of Φ . We are interested in the *adiabatic* evolution of the trapped solutions, as both κ_0 and τ are slow functions of z [see Eq. (20)]. These adiabatic solutions play an important role in the autoresonance phenomenon, and therefore, we shall assume that (21) is satisfied, i.e., τ is of $O(\epsilon)$.

The detailed analysis of the evolution of the trapped, adiabatically varying solutions in our system can be done similarly to Ref. 14. We write $\gamma = \gamma_0(z) + \delta\gamma$, $\Phi = \Phi_0(z) + \delta\Phi$, where as stated above, $\delta\gamma$ is viewed as being of $O(\epsilon^{1/2})$. Assuming the trapped regime [$d\Phi/dz$ is small and of $O(\epsilon^{1/2})$], we set $[k(z) - \Omega(z, \gamma_0)]_{z=z_0} = 0$ at the initial integration point $z = z_0$. Next, we consider two cases depending on the amplitude of the oscillating solution for $\delta\Phi$. The first case corresponds to a situation when $\delta\Phi$ may be of $O(1)$, but $\Phi_0 \ll 1$. We shall define this case as the *loose trapping mode* and observe that it is this situation that was analyzed in almost all the studies of the autoresonance described in the Introduction. The second trapped

regime of interest is the *strong trapping mode*, when $\delta\Phi \ll 1$, but Φ_0 may be of $O(1)$. We shall see later that the strong trapping mode is especially important in our accelerator applications.

We start the analysis from the loose trapping mode. By separating the "fast" oscillating and the slow evolutionary parts in (19), in this case, we obtain

$$\frac{d\gamma_0}{dz} = \epsilon f_0 \sin \Phi_0 \langle \cos(\delta\Phi) \rangle_{\text{av}},$$

$$\kappa_0^2 \sin \Phi_0 \langle \cos(\delta\Phi) \rangle_{\text{av}} = \tau, \quad (22)$$

$$\frac{d(\delta\gamma)}{dz} = \epsilon f_0 \sin(\delta\Phi),$$

$$\frac{d^2(\delta\Phi)}{dz^2} + \kappa_0^2 \sin(\delta\Phi) = 0,$$

where f_0 is evaluated at γ_0 , and the averaging is over a period of the "fast" oscillations of the "pendulum." The knowledge of $\delta\Phi$ from the fourth equation in (22) allows, similarly to the $k=0$ case, to find $\delta\gamma$ by using the third equation in (22). The inequality $\Delta\gamma/\gamma_0 \ll 1$ then yields the condition similar to (16), while the neglect of the term with $\cos \Phi$ in the second equation in (19) yields again the condition (17). The knowledge of $\delta\Phi$ also allows us to calculate $\langle \cos(\delta\Phi) \rangle_{\text{av}}$, which, in turn, can be used to find the slow variables in the system. For example, the second equation in (22) yields $\sin \Phi_0 = (dk/dz) \times [\kappa_0^2 \langle \cos(\delta\Phi) \rangle_{\text{av}}]^{-1}$, which, since $\Phi_0 \ll 1$, leads to the desired dimensionless adiabaticity criterion in our system:

$$|\tau| \ll \kappa_0^2 \langle \cos(\delta\Phi) \rangle_{\text{av}}. \quad (23)$$

Note that this is a stronger condition than (21). Finally, the substitution of $\sin \Phi_0$ in the first equation in (22) yields $(d/dz)[k(z) - \Omega(\gamma_0, z)] = 0$, and, as a result, $k(z) - \Omega(\gamma_0, z) \equiv 0$, since this relation is satisfied at the initial position z_0 , by assumption. More explicitly,

$$\omega_0 - k(z) U_{x0}(z) - \Omega_0/\gamma_0(z) \equiv 0, \quad (24)$$

which is the gyroresonance condition (1) evaluated at γ_0 continuously (at all z). In conclusion, we have shown that in the loose trapping mode, under the conditions (16), (17), and (23) (referred to as the *autoresonance conditions*, in the following) the electron, *on average*, remains in gyroresonance, *despite* the variation of the systems parameters. This self-tendency of the system to stay in resonance continuously for an arbitrary rate of variation of the parameters (wave vector k in our example), when this rate is sufficiently slow, constitutes the main salient feature of the *autoresonance* phenomenon.

Now we consider the strong trapping mode. The necessary conditions for this case are (16), (17), and (21), while the analysis is very similar to that of the loose trapping mode case. Instead of (22), we now have

$$\begin{aligned}
d\gamma_0/dz &= \epsilon f_0 \sin \Phi_0, \\
\kappa_0^2 \sin \Phi_0 &= \tau, \\
\frac{d(\delta\gamma)}{dz} &= \epsilon f_0 (\cos \Phi_0) (\delta\Phi), \\
\frac{d^2(\delta\Phi)}{dz^2} + \kappa_0^2 (\cos \Phi_0) (\delta\Phi) &= 0.
\end{aligned} \tag{25}$$

The second equation in (25) yields $\sin \Phi_0 = \tau/\kappa_0^2$, which, after the substitution into the first equation in (25), again gives $(d/dz)[k(z) - \Omega(\gamma_0, z)] = 0$, so that the autoresonance relation (24) is also satisfied in the strong trapping mode. Now, we can view (24) as defining γ_0 , while the knowledge of γ_0 and Φ_0 allows us, in principle, to find the oscillating components $\delta\gamma$ and $\delta\Phi$ by using the last two equations in (25), and this completes the solution for the strong trapping mode.

Let us compare the loose and the strong trapping autoresonant solutions. The loose trapping mode allows a *broader* range of initial conditions, since it does not require precision in choosing the relative phase Φ . Nevertheless, initially we still have to match the gyroresonance condition (1) within $O(\epsilon^{1/2})$ accuracy. In addition, the strong adiabaticity condition (23) restricts the allowed rate of variation of k , which, in turn, lowers the acceleration gradient in the system (see below). From this point of view, the strong trapping mode is preferable in accelerator applications, since it requires a more relaxed adiabaticity condition (21). On the other hand, at this point, it seems that the required accuracy on the initial conditions in satisfying both the gyroresonance and the phase matching ($\Phi \approx \Phi_0$), makes the strong trapping regime impractical. Nevertheless, by studying the problem of the trapping into the resonance in the next section, we shall show how the strong trapping mode can still be reached in experiments.

We conclude this section by making several remarks regarding the theory of the ridged tapering acceleration scheme based on the spatial tailoring of the k vector (the gyroresonance accelerators utilizing the rigid tapering idea⁴⁻⁷ were all based on the tapering of the guide magnetic field). This scheme is based on a different solution of the same system (19) of equations for γ and Φ , where the local gyroresonant condition $k(z) - \Omega = 0$ is added as an additional constraint, yielding a certain (rigid) dependence $k = k(z)$. Suppose that we have found such $k(z)$. Then, the second equation in (19) shows that the spatial evolution of the system proceeds from the *phase trapping stage* in which $\Phi \rightarrow \pi/2$. After this stage, the first equation in (19) transforms into $d\gamma/dz = \epsilon U_\perp / U_z$. Thus, we obtain the *accelerating* solution. The equation for γ must be solved in combination with Eq. (18) and the exact gyroresonance condition $k(z) \equiv \Omega(z, \gamma)$, so we have a complete set of equations [say for $\gamma(z)$, $u_z(z)$, and $k(z)$]. However, the analytic solution of this *second-order* differential system is difficult, and only the luminous case (characterized by an additional constant of motion) yields a simple dependence $\gamma = \gamma(z)$. More generally, one must base the analysis on numerics. In contrast, from the point of view of complex-

ity, the autoresonance scheme described above is similar to the luminous case, in the sense that one needs to solve Eq. (18) only, subject to the gyroresonance condition (24), since the k -vector dependence is *given*. This task can be easily accomplished analytically (see Sec. VI).

V. TRAPPING INTO THE RESONANCE

In this section we discuss the question of whether the autoresonant regime can still be reached, if, initially, the gyroresonance condition (1) is *strongly* violated. The general answer to this question is negative, since the trapping into the resonance requires passing the separatrix between the trapped and untrapped trajectories in the phase plane of our "effective" nonlinear pendulum [see Eq. (20)]. Such a crossing, however, is a nonadiabatic phenomenon and, usually cannot be achieved by adiabatically varying the parameters of the system. However, there exists a special case in our system, when the crossing is not only allowed, but is, actually, *guaranteed* in our system. Furthermore, this case automatically results in the strong trapping mode (see above), advantageous for the acceleration purposes. This situation corresponds to the case when one initially starts outside the resonance, but with a sufficiently *small* perpendicular velocity ($u_\perp \ll u_z$). Note that such entrance conditions are natural in the case of a beam injected into the system along the guide magnetic field. In order to demonstrate the strong trapping into the resonance for small initial u_\perp , we shall use the results of Ref. 14, where a similar effect was noticed for the first time. For the analogy, we rewrite our system of equations in a different form.

We proceed from the identity $\gamma^2 = 1 + (\gamma u_\perp)^2 + (\gamma u_z)^2$, which, upon the differentiation with respect to z and use of the first and third equations in (9), yields the equation for u_\perp :

$$\frac{du_\perp}{dz} = \epsilon u_\perp (\gamma u_z)^{-1} \left(1 - u_\perp - \frac{u_z}{u_p} \right) \sin \Phi. \tag{26}$$

Combining this result with the second equation in (19) and writing these equations to the lowest significant order with respect to u_\perp / u_z , we arrive at the system

$$\begin{aligned}
\frac{du_\perp}{dz} &= \epsilon h u_\perp \sin \Phi, \\
\frac{d\Phi}{dz} &= k(z) - \Omega + \left(\frac{\epsilon h}{u_\perp} \right) \cos \Phi,
\end{aligned} \tag{27}$$

where $h \equiv (\gamma u_z)^{-1} (1 - u_\perp / u_p)$, and, to the lowest order, both Ω and h are constant. It is the $1/u_\perp$ singularity in the second equation in (27), which if initially $\epsilon h / u_\perp \gg 1$ [note that this is the opposite to what is required by (17)], leads to the *strong trapping* into the resonance when $k(z) - \Omega_\perp$ passes zero, due to the variation of k . This phenomenon was demonstrated in Ref. 14, where the system of form (27) was analyzed in detail in application to the problem of mode conversion. We, therefore, come to a similar conclusion in our case, demonstrating another salient feature of our approach, i.e., the possibility of injecting the electron beam neither in the vicinity of the gyrores-

onance, nor with an exact relative phase. The beam, eventually, will be strongly trapped into the resonance, provided¹⁴ (a) one starts sufficiently far from the resonance; (b) the transverse velocity of the beam is initially sufficiently small; and (c) $k(z) - \Omega$ passes zero due to the variation of k in the direction of propagation of the beam. After the trapping,¹⁴ u_{\perp} increases rapidly, so that the initial inequality $\epsilon\hbar/u_{\perp} \gg 1$ is reversed, and the system enters the autoresonance phase of the evolution.

VI. AUTORESONANT ACCELERATION AND NUMERICAL EXAMPLES

The possibility of using the spatial autoresonance for acceleration is based on the described above self-tendency of the system to adjust the average value γ_0 to that of the k vector in order to satisfy the autoresonance relation (24) continuously. Therefore, one of the directions of variation of k leads to the average *acceleration*. Let us consider this question in more detail. We proceed from Eq. (18), which, to the lowest order in ϵ yields

$$\frac{d\gamma_0}{dz} = \frac{u_p d(\gamma_0 U_{z0})}{dz}. \quad (28)$$

On the other hand, the autoresonant relation (24) gives

$$\gamma_0 U_{z0} = u_p (\gamma_0 - \Omega_0/\omega_0). \quad (29)$$

By substituting this expression into the RHS of (28), we obtain

$$(1 - u_p^2) \left(\frac{d\gamma_0}{dz} \right) = \left(\gamma_0 - \frac{\Omega_0}{\omega_0} \right) u_p \left(\frac{du_p}{dz} \right), \quad (30)$$

or, after the integration,

$$\gamma_0(z) = \frac{\Omega_0}{\omega_0} + \left(\frac{\gamma_0 U_{z0}}{u_p} \right)_{z=z_0} \left(\frac{1 - u_p^2(z_0)}{1 - u_p^2(z)} \right)^{1/2}. \quad (31)$$

This simple formula shows that if $u_p(z)$ approaches unity, one has an accelerating solution for either fast ($n < 1$) or slow ($n > 1$) electromagnetic waves.

At this stage, let us illustrate our theory by numerical examples. We shall use the TE₁₁ waveguide mode in these examples, so in the vicinity of the axis of the waveguide the electromagnetic fields are of form (2), and are characterized by the local dispersion relation

$$\omega_0^2 = k^2 + (1.841/R)^2, \quad (32)$$

where R is the waveguide radius. We shall assume that the variation of k with z is the result of an increase of this radius along the axis of the waveguide. We shall also assume that the variation of the radius is sufficiently slow, and does not lead to a significant reflection of the electromagnetic wave or introduction of other waveguide modes in the system. This requires satisfaction of the dimensionless adiabaticity condition $k^{-2} |dk/dz| \ll 1$, i.e., the operation of the accelerator sufficiently far from the cutoff. Figures 2(a) and 2(b) show the results of the numerical solution of Eqs. (9) for γ , u_z , and u_{\perp} in the case of $\omega_0 = 1.4 \text{ cm}^{-1}$ and two values of ϵ at the initial integration point ($z = z_0 = 0$), i.e., 0.02 cm^{-1} (solid lines) and 0.04 cm^{-1}

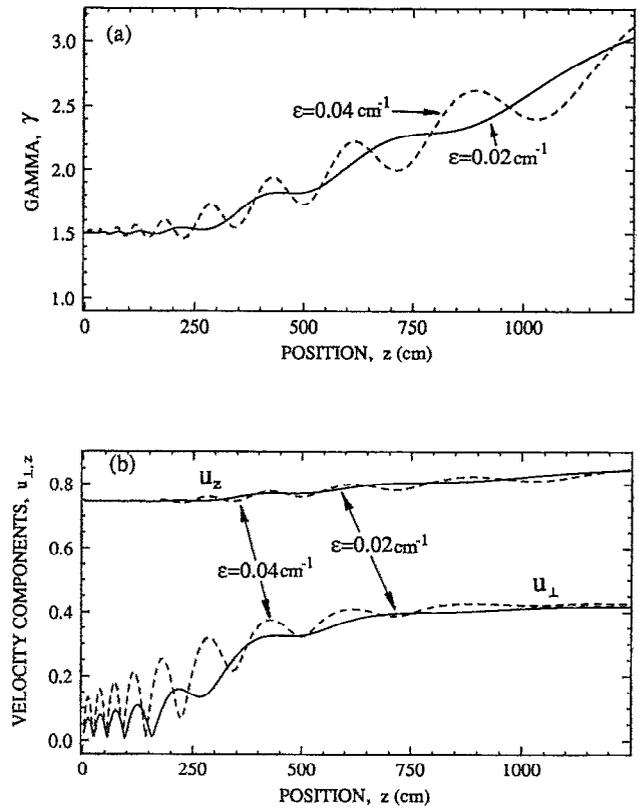


FIG. 2. The z dependence of (a) γ , and (b) $u_{\perp, z}$ for two values of the normalized amplitude ϵ of the TE₁₁ electromagnetic field in the case of a linearly increasing waveguide radius $R = R_0(1 + \alpha z)$. The linear growth of the average value of γ and its oscillations around this average are the signatures of the spatial autoresonance in the system.

(dashed lines). We have assumed a linear dependence $R = R_0(1 + \alpha z)$, with $\alpha = 0.002 \text{ cm}^{-1}$ and $R_0 = 1.8 \text{ cm}$. Also, initially (at $z = 0$), we have set $\gamma = 1.5$, $u_{\perp} = 0$, and assumed $\epsilon(z) = \epsilon(0)R/R_0$. The guide magnetic field was constant ($B_0 = 1.23 \text{ kG}$), so, at $z = 0$, Ω_0/γ was equal to $0.7(\omega_0 - ku_z)$, i.e., significantly off the gyroresonance. We see in Fig. 2(a) that, for both values of $\epsilon(0)$, the spatial dependence of γ is similar. In the initial evolution stage $z < 250 \text{ cm}$, there is no average change in the electron energy and γ oscillates around its initial value of 1.5. These initial oscillations are characteristic of the untrapped region of the phase space of Eq. (20). However, when, due to the variation of the k vector, we approach the resonant position $z \approx 250 \text{ cm}$, the trapping into the resonance takes place, which can be seen in the characteristic evolution of u_{\perp} in the trapping phase in Fig. 2(b), i.e., the fast oscillations followed by a sudden rapid growth of u_{\perp} as the system enters the autoresonant phase.¹⁴ We can see in Fig. 2(a) that, for $z > 250 \text{ cm}$, the factor γ , while still oscillating, increases *linearly* with z on average. Both this linear growth and the oscillations are the characteristic signatures of the autoresonance in our case. For instance, we observe that the average spatial growth rate of γ is independent of the amplitude of the electromagnetic wave, as the particle, on average, preserves the gyroresonance condition (which is independent of ϵ). The amplitude of the

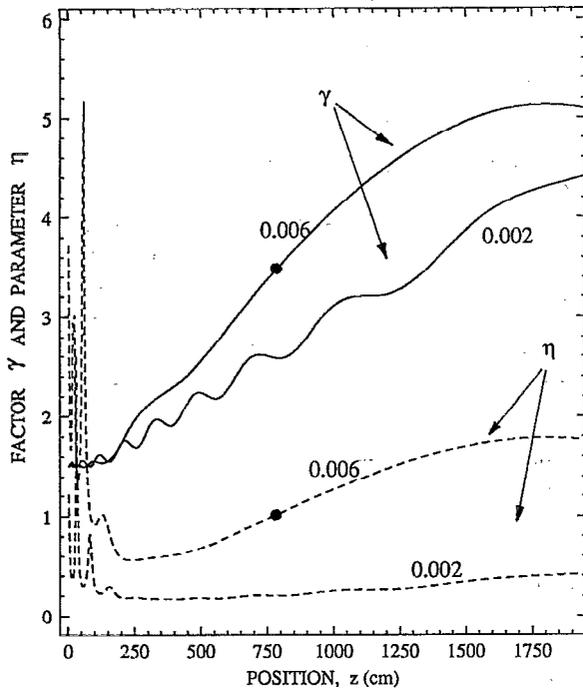


FIG. 3. The z dependence of the factor γ (solid lines), and the trapping parameter η (dashed lines) for two values of the slope $\alpha = [R(0)]^{-1} [dR(z)/dz]$ (shown in units of cm^{-1}) of the spatial increase of the waveguide radius. The regions with $\eta < 1$ correspond to the strong trapping autoresonant interaction mode. At the position (shown by full circles) where $\eta = 1$, the beam detrap from the gyroresonance.

oscillations of γ , in contrast, depends on ϵ and scales as $\sim \epsilon^{1/2}$. We can also explain the linear dependence of γ_0 with position. Indeed, the dispersion equation (32) yields

$$1 - u_p^2 = A^2 / (R^2 - A^2), \quad (33)$$

where $A = 1.841/\omega_0$. Therefore, for $(R/A)^2 \gg 1$, we can rewrite (31) as

$$\begin{aligned} \frac{\gamma_0(z)}{\gamma_0(z_0)} &= \frac{\Omega_0}{[\gamma_0(z)\omega_0]} + \left(\frac{U_{z0}}{u_p}\right)_{z=z_0} \frac{R(z)}{R(z_0)} \\ &= 1 + \left(\frac{U_{z0}}{u_p}\right)_{z=z_0} \left(\frac{\alpha R_0}{R(z_0)}\right) (z - z_0), \end{aligned} \quad (34)$$

where, now, z_0 corresponds to the position where the gyroresonance condition was satisfied first.

Equation (34) shows that the acceleration gradient increases with α . We are illustrating this prediction in Fig. 3, showing γ vs z (solid lines) for the same initial conditions and parameters as in Fig. 2 [$\epsilon(0) = 0.04 \text{ cm}^{-1}$ case], but for two values of $\alpha = 0.002, 0.006 \text{ cm}^{-1}$, and arranging Ω_0 , for each α , so that the beam passes the gyroresonance at $z \approx 150 \text{ cm}$ (the corresponding values of B_0 are 1.37 and 1.12 kG). We see in the figure that the initial average growth of γ in the autoresonant region is linear with z and the slope increases with α . However, we can also see that at $\alpha = 0.006 \text{ cm}^{-1}$ the growth of γ slows down at larger z and later saturates. The reason for the change in the character

of the spatial dependence of γ is the violation of the condition (21). Let us consider this problem in more detail.

First, we use the autoresonance relation (24), which, on differentiation, yields

$$\tau = \left(\frac{\partial \Omega(\gamma_0, z)}{\partial \gamma_0}\right) \left(\frac{d\gamma_0}{dz}\right). \quad (35)$$

By using (35) and $d\gamma_0/dz$ from (30), we can rewrite condition (21) as

$$[\epsilon U_{\perp} (1 - u_p^2)]^{-1} (\gamma_0 U_{z0}) \left(\frac{du_p}{dz}\right) < 1, \quad (36)$$

which, for the dispersion relation (32), gives

$$\eta \equiv \alpha \gamma_0 U_{z0}^2 u_p [\epsilon(0) U_{\perp 0}]^{-1} < 1. \quad (37)$$

The z dependence of the parameter η for the two cases in Fig. 3 is shown in the same figure by the dashed lines. It can be seen in the figure that, in both cases, $\eta < 1$ in the autoresonant (linear growth) regime. However, when in the $\alpha = 0.006 \text{ cm}^{-1}$ case, due to the increase of γ , η increases and passes the position (shown by the full circles in Fig. 3), where $\eta = 1$, we observe the slowing down of the growth of γ . This is the result of the *detraping* of the electrons, leading to the departure from the gyroresonance, and, finally, to the saturation. It is interesting to point out that, since $\eta = 1$ at the detraping point, according to (25), $\Phi_0 = \pi/2$, while the autoresonance condition is still satisfied. But these are precisely the conditions for the rigid tapering acceleration scheme (see above). Therefore, one can use the autoresonance mechanism as a method for preparing the beam for use in the second stage (beyond the point where $\eta = 1$), rigid tapering acceleration scheme. In our example, this would require a certain z dependence of the k vector or/and of the guide field, which would prevent the detrimental dephasing from the gyroresonance beyond the detraping point. If, however, we limit ourselves to the autoresonance scheme, it is important to find the optimal value α_{opt} of α for a given acceleration distance z . This α_{opt} can be defined to give unity for the parameter η at the end of the acceleration interval, i.e., $\alpha_{\text{opt}} = \epsilon(0) U_{\perp 0} (\gamma_0 U_{z0}^2 u_p)^{-1}$. With this definition, and for $z \gg z_0$ and $u_z, u_p \approx 1$, Eq. (34) yields the following "optimal" scaling law in our example:

$$\gamma_0(z) \approx \gamma_0(0)/2 + \sqrt{\gamma_0^2(0)/4 + U_{\perp}(z)\epsilon(0)z}. \quad (38)$$

This scaling is weaker than that in linear accelerators or in the rigid tapering scheme for the idealized luminous case. Nevertheless, this difference is less pronounced at lower energies, which, in combination with such advantages of the method as the stability with respect to initial conditions, freedom in choosing the rate of the tapering and the possibility of starting off the gyroresonance, makes the strong trapping autoresonant accelerator an attractive possibility.

VII. CONCLUSIONS

(i) We have analyzed a linear, spatial autoresonance cyclotron acceleration (SACA) scheme based on the self-

tendency of the particles to stay in gyroresonance, despite the variation of the systems parameters (the k vector of the wave in our case).

(ii) We have found the necessary conditions for operation in both the loose and the strong trapping autoresonant modes and shown that the strong trapping mode is preferable, since it requires a less restrictive adiabaticity condition. Furthermore, the method of entering the strong trapping mode via the efficient trapping into the gyroresonance for sufficiently small initial transverse beam velocities was analyzed and illustrated in numerical examples.

(iii) In comparison with other cyclotron resonance acceleration methods, the SACA provides (a) DC operation (in contrast to the GYRAC concept requiring pulsed guide fields), and (b) stability with respect to initial conditions and rate of variation of system parameters (in contrast to rigid chirping cyclotron acceleration schemes). Because of this enhanced stability the SACA may be more tolerant to other destabilizing factors (such as high currents and radial dependence of the driving electromagnetic fields) than other acceleration schemes. We shall analyze these problems in our future work.

(iv) We have studied the problem of the detrapping from the gyroresonance due to the violation of the adiabaticity condition. This is the main source for the saturation in the SACA. However, it was shown that, at the detrapping position in the SACA, the electron beam is ideally

prepared for the use in the rigid tapering acceleration scheme, and, therefore, a hybrid accelerator utilizing both methods is feasible.

ACKNOWLEDGMENTS

The author acknowledges the PMI group management and staff of the Ecole Polytechnique for providing the excellent environment for this research during the sabbatical leave from the Hebrew University of Jerusalem. The author is especially grateful to Dr. M. Bacal for her extensive personal and administrative help and to Dr. J. M. Buzzi for useful scientific discussions.

- ¹C. S. Roberts and S. J. Buchsbaum, *Phys. Rev. A* **135**, 381 (1964).
- ²H. R. Jory and A. W. Trivelpiece, *J. Appl. Phys.* **39**, 3053 (1968).
- ³V. Ya. Davydovskii, *Sov. Phys. JETP* **16**, 629 (1963).
- ⁴P. Sprangle, L. Vlahos, and C. M. Tang, *IEEE Trans. Nucl. Sci.* **NS-30**, 3177 (1983).
- ⁵A. Loeb and L. Friedland, *Phys. Rev. A* **33**, 1828 (1986).
- ⁶C. Chen, *Phys. Fluids B* **3**, 2933 (1991).
- ⁷R. Shpitalnik, C. Cohen, F. Dethan, and L. Friedland, *J. Appl. Phys.* **70**, 1101 (1991).
- ⁸K. S. Golovanivsky, *IEEE Trans. Plasma Sci.* **PS-11**, 28 (1983).
- ⁹O. Gal, *IEEE Trans. Plasma Sci.* **PS-17**, 622 (1989).
- ¹⁰B. Meerson and L. Friedland, *Phys. Rev. A* **41**, 5233 (1990).
- ¹¹B. Meerson, *Phys. Lett. A* **150**, 290 (1990).
- ¹²B. Meerson and S. Yariv, *Phys. Rev. A* **44**, 3570 (1991).
- ¹³G. Cohen and B. Meerson, *Phys. Rev. E* **2**, 967 (1993).
- ¹⁴L. Friedland, *Phys. Fluids B* **4**, 3199 (1992).
- ¹⁵L. Friedland, *Phys. Rev. Lett.* **69**, 1749 (1992).