Passage through resonance and autoresonance in $x^{2n}$-type potentials

E. Nakar and L. Friedland
Racah Institute of Physics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel
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Resonant dynamics of a particle in an $x^{2n}$-type potential driven by an oscillation with adiabatically varying frequency is investigated. It is shown that, under certain conditions, when the driving frequency increases in time and passes the resonance with the unperturbed system, the oscillator phase locks to the drive and, later, this phase locking is sustained, i.e., the system remains in autoresonance. The initial phase locking by passage through resonance is the main ingredient of the transition to autoresonance and comprises the generalization of previous results for nearly parabolic potentials. [S1063-651X(99)16911-3]

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I. INTRODUCTION

Autoresonance (or self-sustained resonance) is a phenomenon taking place when a resonantly driven nonlinear system stays phase locked with adiabatically varying perturbing oscillation (the drive) even if the drive’s frequency (or another system parameter) varies in time. Autoresonance is used in relativistic particle accelerators [1] and other applications, such as, in atomic and molecular physics [2], nonlinear dynamics [3], nonlinear waves [4], fluid dynamics [5], and plasma physics [6]. There exist two scenarios for exciting autoresonance in the system. The first is applying a driving perturbation with initial frequency and phase tuned close to those of the unperturbed nonlinear oscillator, which may or may not be excited initially [2]. In this case, at certain conditions, the oscillator remains in resonance (stays phase locked) at later times, as the driving frequency varies in time. Alternatively [7], one can start out of resonance and slowly pass the resonant point by chirping the driving frequency. Then, there exists a sharp threshold on the drive’s amplitude [5] beyond which the oscillator automatically phase locks with the drive and evolves in autoresonance at later times if the variation of the driving frequency continues. The second autoresonant excitation scheme is preferable in practical implementations, since it does not require fine initial tuning. We shall refer to this scheme as locking by passage through resonance or LPTR. The theory of LPTR and the aforementioned threshold prediction exist for the case, when one proceeds near the equilibrium position (say $x=0$) of the nonlinear oscillator, and, locally, the confining potential has a nearly parabolic form $V \propto x^2$ [5]. In contrast, in the present work, we shall develop the LPTR theory for oscillators having a nonparabolic form ($V \propto x^{2n}, n>1$) near the equilibrium. This family of potentials comprises a model describing transition to a square potential well as $n \to \infty$. We shall also generalize the LPTR theory to studying the phase locking (with the drive) of initially excited oscillators, and discuss the advanced autoresonance stage in these systems.

Our presentation will be as follows. In Sec. II we shall study the LPTR of a particle for $V \propto x^4$ case in detail. The subsequent autoresonant evolution in this case will be studied in Sec. III. Finally, in Sec. IV we shall generalize to the $n>2$ case and present our conclusions.

II. PHASE LOCKING BY PASSAGE THROUGH RESONANCE

Consider a driven oscillator described by the Hamiltonian

$$H(p,x,t) = \frac{1}{2} p^2 + \frac{1}{2n} x^{2n} - \varepsilon \cos \phi(t),$$

where $n$ may have values 2, 3, . . . , while the last term is a perturbation ($\varepsilon \ll 1$) characterized by frequency $\omega(t) = d\phi/dt$, which is a slowly increasing function of time. For simplicity, we shall assume that this time dependence is linear, i.e., $\omega(t) = \omega_0 + \Delta t$ and that all dependent and independent variables ($p, x, t$) and parameters ($\varepsilon, \omega$) are dimensionless. We shall also assume that the oscillator is excited initially (at $t=t_0$) and study the possibility of phase locking via passage through resonance (LPTR) between the drive and the oscillator and subsequent evolution of the system at later times, $t > t_0$. A typical example illustrating LPTR and the autoresonance in this system is presented in Fig. 1(a), where we show the phase space portrait of the solution of

![FIG. 1. Autoresonance in the $V=\frac{1}{4}x^4$ potential well. (a) The spiralling phase-space portrait of autoresonant evolution; (b) The coordinate versus time (curve 1) and the function $\cos \phi(t)-3.5$ (curve 2) representing the driving oscillation. Comparison between the curves illustrates the persisting phase locking in the system despite variation of the driving frequency.](image_url)
Hamilton’s equations \( dp/dt = -x^2 + 2\cos \varphi \), \( dx/dt = p \) in the case \( n = 2 \). We used parameters \( \epsilon = 0.11 \), \( \alpha = 0.005 \), \( \omega_0 = 0 \), (i.e., \( \varphi = \frac{1}{2} \alpha t^2 \)) and initial conditions \( x = 0 \), \( p = 0.372 \) (at \( t = 10 \)) in these calculations. Additional information from the calculations is presented in Fig. 1(b), where we see a part of the evolution of \( x \) (curve 1) in the time interval 200 < \( t < 400 \), as well as the function \( \{ \cos(\varphi(t)) - 3.5 \} \) (curve 2 in the figure) representing the shifted (by \( -3.5 \)) driving perturbation. The phase space portrait in Fig. 1(a) exhibits a spiralling evolution with a slowly increasing averaged energy, while the comparison between the curves in Fig. 1(b) illustrates the phase locking between the solution and the drive despite the variation of the driving frequency. We can also see that the amplitude of these oscillations grows on the average, but also performs slow oscillations around the average. The interference-type pattern in Fig. 1(a) is due to these amplitude oscillations. Our goal is to discuss all stages of evolution in Figs. 1(a) and 1(b) and find the conditions for trapping into resonance and subsequent autoresonance in the system.

The most convenient description of the resonant dynamics in our system is obtained by transforming to action angle variables \( L \) and \( \Theta \) associated with the unperturbed problem described by Hamiltonian \( H_0 = \frac{1}{2} p^2 + 1/2 n x^2 \). In this case, \( H_0 = b_n I^2/(n+1) \), where \[ b_n = (2n)^{1/2} \frac{\Gamma((1/2+1/2n))}{\Gamma(1/2n)} \] These results are shown in Fig. 2(a), where \( n = 1 \), \( n = 2 \) case, i.e., \( \sqrt{1/4} x^2 \), for which (8) becomes

\[ d l/dt = - \epsilon \gamma_n I^2 \sin \Phi, \] (8)

where, for \( n = 1 \), we have shifted the time to remove the constant term in the right-hand side (RHS) in the second equation in (7); \( r_n = 1/(n+1) \), \( q_n = (n-1) r_n \), \( c_n = b_n \) for \( n > 1 \), and \( q_n \) for \( n = 1 \). Note that (8) comprises a Hamiltonian system for the canonical pair \((I, \Phi)\) with the effective Hamiltonian of form

\[ H_{eff} = c_n I^3 + \alpha I - \epsilon \gamma_n I^2 \sin \Phi. \] (9)

Now, for definiteness, we focus on the \( n = 2 \) case, i.e., \( V = \frac{1}{4} x^2 \). For which (8) becomes

\[ d l/dt = - \epsilon \gamma_2 I^2 \sin \Phi, \] (10)

where one sub-
FIG. 2. Autoresonance in two- and (beyond \( t_i \sim 75 \)) single-resonance approximations. (a) The evolution of the action \( I \) and phase mismatch \( \Phi \). The thick line represents the exact resonance condition; (b) the energy in the system versus time. \( H \) is the solution of the exact evolution equations; \( H' \) (the thin line) is calculated by using the results shown in (a) substituted in the approximation (5).

stificates the data shown in Fig. 1(a) and compares it with \( H \) [see (1)] calculated by using the numerical solution of the original evolution equations. We can see in this figure that for \( t_0 < t < 75 \) the agreement is excellent, while for \( t > 75 \), when we switch to the single resonance approximation, continues to be very good.

At this stage we proceed to the question of how the phase locking starts in our system. One can answer this question by viewing \( \omega(t) = \alpha t \) in the Hamiltonian \( H_{\text{eff}} \) [see (9)] as a slowly varying parameter. If this parameter would be fixed, the energy \( H_{\text{eff}} \) would be conserved. The phase space portrait of the system in this case can be analyzed by rewriting (9) (for \( n = 2 \) case) as

\[
G(I) = \cos \Phi = (\epsilon \gamma_2)^{-1} \left( c_2 I - \omega I^{2/3} - \frac{H_{\text{eff}}}{I^{1/3}} \right). \tag{11}
\]

By studying the RHS of the last equation one finds that for sufficiently small \( \omega \) the phase space portrait of the system changes its character depending on whether \( H_{\text{eff}} \) is positive or negative. This is illustrated in two examples in Figs. 3(a) and 3(b). Figure 3(a) shows \( G = G(I) \) for \( \omega = 0.4, \epsilon = 0.11 \) and three values \( H_{\text{eff}} = -2 \times 10^{-3}, 0.2 \times 10^{-3} \) (curves 1, 2 and 3, respectively). We can see from the figure that for \( H_{\text{eff}} = 2 \times 10^{-3} \) the phase mismatch \( \Phi \) grows monotonically \( [ -1 \leq \cos \Phi \leq +1 ] \), while the action oscillates in the interval \( I_1 \leq I \leq I_2 \) (between the corresponding full circle dots in the figure). In contrast, for \( H_{\text{eff}} = -2 \times 10^{-3} \) the phase mismatch is bounded and \( \cos \Phi \) oscillates between \( +1 \) and \( \cos (\Delta \Phi) \), i.e., \( \Phi(\text{mod } 2 \pi) \) oscillates between \( \pm \Delta \Phi \), while the oscillations of the action remain in nearly the same limits \( I_3 \leq I \leq I_4 \). Figure 3(b) presents the same case as Fig. 3(a), but for a larger value of \( \omega = 0.8 \). One can see that in contrast to Fig. 3(a), the minimum of \( G(I) \) at \( H_{\text{eff}} = 0 \) now lies below \( -1 \). Consequently, when one slowly passes from small positive to small negative values of \( H_{\text{eff}} \), the system remains detrapped, while the action oscillates in almost the same interval \( I_1 \approx I_3 \approx I_2 \). We shall see below that, at later times in this case, the minimum value of \( G \) increases until it passes the value of \( -1 \), while, later, for some initial values of \( \Phi, G \) decreases again, after the quasi-particle passes to the second detrapped region [denoted by two square dots in Fig. 3(b)].

Now we return to our real system, where the parameter \( \omega(t) = \alpha t \) is a slowly varying function of time. In this case \( H_{\text{eff}} \) also is a slow function of time. Since

\[
dH_{\text{eff}}/dt = -\alpha I, \tag{12}
\]

\( H_{\text{eff}} \) decreases monotonically. As a result, if one starts at a positive value of \( H_{\text{eff}} \) and the rate of change of \( H_{\text{eff}} \) is sufficiently slow (i.e., for sufficiently small \( \alpha \)), on the basis of the discussion above, we expect phase trapping to occur at the time when \( H_{\text{eff}} \) passes zero, provided that the minimum of \( G(I) \) at this time is above \( -1 \). Note that this argument is independent of the initial value of the phase mismatch and, therefore, the initial positiveness of \( H_{\text{eff}} \) and the first occurrence of the minimum in \( G(I) \) above \( -1 \) are the necessary conditions for phase trapping in the system. We demonstrate these conclusions in Fig. 4(a) showing the evolution of the phase mismatch \( \Phi \) and of \( H_{\text{eff}} \) in the example presented earlier in Fig. 1. As expected, phase trapping occurs at the moment when \( H_{\text{eff}} \) passes zero. In contrast, in Fig. 4(b), we start with a larger initial value of the action, \( I = 0.46 \) (larger initial energy) and switch to the single resonance approximation [Eqs. (11)] at later time, \( t_0 = 120 \). Here, the minimum of \( G \) is less than \( -1 \) when it appears first during the evolution and, consequently, there is no phase trapping in this case. Finally, we estimate the phase trapping time. The latter de-
The minimum of \( G \) occurs first above \(-1\). The phase trapping proceeds as \( H_{eff} \) passes zero. (b) Larger initial energy. The minimum of \( G \) occurs first below \(-1\) and there is no phase trapping in this case.

![Figure 4: Energy \( H_{eff} \) and phase mismatch \( \Phi \) versus time in passage through resonance. (a) Sufficiently small and positive initial \( H_{eff} \). The minimum of \( G \) occurs first above \(-1\). The phase trapping proceeds as \( H_{eff} \) passes zero. (b) Larger initial energy. The minimum of \( G \) occurs first below \(-1\) and there is no phase trapping in this case.](image)

FIG. 5. Action versus phase mismatch during phase trapping via passage through resonance. The conservation of the areas \( A_i \) in the figure illustrates the adiabatic invariance of \( J = (2\pi)^{-1} \int I^s d\Phi \) during the evolution from the detrapped (areas \( A_{1,2,3} \)) to trapped (area \( A_3 \)) state of the system. The arrows show the direction of the time evolution of the system.

where \( I^s = I^s(H_{eff}, \omega, \Phi) \) is the solution of (11) for \( I \). We illustrate the invariance of \( J \) in Fig. 5, showing the \((I, \Phi)\) portrait of evolution in the system passing from the detrapped to trapped autoresonant state. We used the same initial conditions and parameters as in Figs. 1 and 2, but \( \alpha = 0.001 \) and \( t_r = 275 \). The invariance of \( J \) is seen in the figure as the preservation of the areas \( A_{1,2,3} \) inside the parts of the portrait shown by thick lines and corresponding to single oscillation periods at different times before (areas \( A_{1,2} \)) and after (areas \( A_3 \)) the trapping. Now, if at \( t = t_0 \), \( J = J_0 \), and since \( \omega = \alpha t \), the expected trapping time \( t_r \) is given by \( J_r = \alpha t_r = J_0 \). In addition to \( J_r(s) \) (s being defined as \( s = \alpha t \)), additional two universal functions evaluated at \( H_{eff} = 0 \) are important in studying the evolution of the system beyond the trapping point (see below). The first is the value of \( I^s \) averaged over an oscillation, i.e.,

\[
I^s_r(s) = T^{-1} \int_0^T I^s dt,
\]

where one averages over the period of oscillation \( T(s) = (\partial \lambda / \partial H_{eff})^{-1} \) of the system for fixed values of \( H_{eff} = 0 \) and \( \omega = s \). The second function, denoted by \( I^s_{mr}(s) \), is the value of \( I^s \) at which the function \( G \) [see (11)] at \( H_{eff} = 0 \) and \( \omega = s \) has a minimum (recall that for trapping this minimum must lie above \(-1\)). As defined, the three functions, \( J_r(s) \), \( I^s_r(s) \), and \( I^s_{mr}(s) \), depend on two parameters, \( \varepsilon \) and \( \alpha \). Nonetheless, the renormalized functions \( \hat{J}_r = (\varepsilon \gamma_2)^{-1} J_r \), \( \hat{I}^s_r = (\varepsilon \gamma_2)^{-1} I^s_r \), and \( \hat{I}^s_{mr} = (\varepsilon \gamma_2)^{-1} I^s_{mr} \) depend on a single parameter \( \hat{\alpha} = (\varepsilon \gamma_2)^{-1/3} \alpha \), via the renormalized variable \( \hat{s} = \alpha t \). We have calculated \( \hat{J}_r(\hat{s}) \), \( \hat{I}^s_r(\hat{s}) \), and \( \hat{I}^s_{mr}(\hat{s}) \) numerically and show these functions in Fig. 6. As described earlier, \( \hat{J}_r(\hat{s}) \) allows to find the trapping time \( t_r \). On the other hand, we observe in the figure that \( \hat{I}^s_r(\hat{s}) > \hat{I}^s_{mr}(\hat{s}) \). This leads to the conclusion that, after the trapping, the minimum value of \( G = \cos(\Phi) \) increases towards \(+1\), i.e., the amplitude \( A\Phi = \cos^{-1}(\min G) \) of the phase mismatch oscillations of the trapped state around \( \Phi = 0 \) (mod \( 2\pi \)) decreases. To prove this statement, we denote \( \min G = G(\hat{I}^s_{mr}) = G^* \) and, by direct differentiation in the vicinity of the trapping point, obtain

\[
dG^*/dt = (\alpha \varepsilon \gamma_2)(I^s_{mr})^{-1/3}(I - I^s_{mr}).
\]
This equation shows that $G^\ast$ increases monotonically if $I-I_{mr}^\ast$ is positive, as in the detrapped region between the full circle dots in Fig. 3(b). Similarly, $G^\ast$ decreases monotonically when the particle is in the second detrapped region [between two square in Fig. 3(b)]. The transition between these two types of evolution corresponds to the case when, for some initial $\Phi$, the particle passes from one detrapped region to another and, thus, remains detrapped after passing through resonance. In contrast, if the particle is trapped (either initially, or when $G^\ast$ appears first above $-1$), $I$ oscillates around $I_{mr}^\ast$ and so does $G^\ast$, according to (15). At the same time, we find that, in averaging over a single oscillation,

$$d\langle G^\ast\rangle/dt=(\alpha/3)\langle G^\ast\rangle(I_{mr}^\ast)^{-4/3}(I^\ast-I_{mr}^\ast)>0,$$  \hspace{1cm} (16)

e.g., indeed, the amplitude $\Delta \Phi$ of the system phase mismatch oscillations decreases after the trapping. Similarly, one shows that

$$d\langle I_{mr}^\ast\rangle/dt=(\alpha/3)G^\ast(I_{mr}^\ast)^{-4/3}(I^\ast+2I_{mr}^\ast),$$  \hspace{1cm} (17)

where $G^\ast>0$ is the second derivative of $G$ at the minimum. Thus, the minimal action value increases after the trapping. When $\langle I_{mr}^\ast\rangle$ grows and the amplitude $\Delta \Phi$ of the mismatch oscillations becomes sufficiently small (say $\Delta \Phi<\pi/6$ for definiteness), the systems enters the advanced autoresonant evolution stage, which we consider in the following section.

### III. ADVANCED AUTORESONANT EVOLUTION

At this point we assume that during the phase locking process described above, one reaches a stage, where $\Phi(mod\ 2\pi)<\pi/6$, i.e., one can replace $\cos \Phi=1$ in the second equation in (10), yielding a simplified system:

$$dI/dt=-\varepsilon\gamma_2 I^{1/3}\sin \Phi,$$  \hspace{1cm} (18)

$$d\Phi/dt=(4/3)c_2 I^{1/3}-\alpha-(\varepsilon\gamma_2/3)I^{-2/3}.$$

Then, we seek solutions of (18) in the form $I=\tilde{I}+\delta I$ and $\Phi=\tilde{\Phi}+\delta \Phi$, where $\tilde{I}$ and $\tilde{\Phi}$ are defined by

$$(4/3)c_2^{1/3}I^{1/3}-\alpha-(\varepsilon\gamma_2/3)^{1/3}I^{-2/3}=0,$$

$$\frac{\alpha}{R}-\varepsilon\gamma_2 I^{1/3}\sin \tilde{\Phi}=0,$$  \hspace{1cm} (19)

where

$$R=\alpha(d\tilde{I}/dt)^{-1}=(4/9)c_2^{1/3}I^{-2/3}+(2/9)\varepsilon\gamma_2 I^{-5/3}.$$  \hspace{1cm} (20)

Note that $\tilde{\Phi}$ can be defined this way only if $\varepsilon\gamma_2 I^{1/3}>\alpha/R$. Furthermore, both $\tilde{I}$ and $R$ are varying monotonically with $t$ ($\tilde{I}$ grows, while $R$ decreases). Now, by assuming that $|\delta I|\ll\tilde{I}$ and $|\delta \Phi|\ll \pi$, we linearize in (18), yielding

$$d(\delta I)/dt=-\varepsilon\gamma_2 I^{1/3}\delta \Phi,$$

$$d(\delta \Phi)/dt=R\delta I.$$

Equations (21) comprise a Hamiltonian system with the Hamiltonian

$$\delta H=\frac{1}{2}R(\delta I)^2+\frac{1}{2}\varepsilon\gamma_2 I^{1/3}(\delta \Phi)^2,$$  \hspace{1cm} (22)

which describes linear oscillations of $\delta \Phi$ and $\delta I$ with a slowly varying frequency $\nu=-(\varepsilon\gamma_2 I^{1/3}/R)^{1/2}$. We observe that if $\delta I_{max}$ and $\delta \Phi_{max}$ are the amplitudes of these oscillations, then the product $(\delta \Phi_{max})(\delta I_{max})$ is an adiabatic invariant as long as $\nu^{-2}[d\nu/dt] \ll 1$. Therefore, since from (21), $\nu\delta I_{max}=(\varepsilon\gamma_2 I^{1/3}/R_{max})$, we have $(\delta I_{max})^2\propto R^{-1/2}I^{1/6}$, so $\delta I_{max}/I\propto R^{-1/4}I^{-1/12}$. Similarly, $\delta \Phi_{max}/I\propto R^{1/3}I^{-1/12}$. These results show that both $\delta I_{max}/I$ and $\delta \Phi_{max}$ decrease with time under adiabatic conditions and, therefore, the adiabatic condition $\nu^{-2}[d\nu/dt] \ll 1$ is sufficient for sustaining the autoresonance beyond the trapping stage. Let us discuss this condition in more detail. After some algebra, we find

$$\nu^{-2}[d\nu/dt]<C[I^{1/6}+2(\varepsilon\gamma_2 c_2)I^{-7/6}],$$  \hspace{1cm} (23)

where $C=(9/16[\alpha/(\varepsilon\gamma_2)]^2(\varepsilon\gamma_2/c_2)^{3/2}$. Since the right hand side in (23) decreases in time, sufficiently small value of $\alpha$ guarantees satisfaction of the adiabaticity condition at all times. Finally, in order to have a slowly varying quasi-steady state defined in (19) one must have $\varepsilon\gamma_2 I^{1/3}/\delta I<1$, or

$$\alpha(\varepsilon\gamma_2 R I^{1/3})^{-1}<1.$$  \hspace{1cm} (24)

This condition brakes for $t$ large enough since $RI^{1/3}$ decreases with time. One can get a good estimate for the braking time $t_b$ by assuming $\tilde{I}_b=\tilde{I}(t_b)\gg 1$. In this case $RI^{1/3} \approx (16/27)c_2^2(\alpha)^{-1}$ and, therefore,

$$t_b \approx (16/27)(\varepsilon\gamma_2/c_2)(c_2/\alpha)^2.$$  \hspace{1cm} (25)

We see that $t_b$ and, consequently, the maximum amplitude of autoresonantly excited oscillator can be controlled by varying the rate of variation $\alpha$ of the driving frequency. Note also that since $\tilde{I}_b=(4\varepsilon\gamma_2 c_2^3(\alpha)^3$, our assumption $\tilde{I}\gg 1$ is valid for $\alpha$ small enough.

We illustrate our predictions in Fig. 7 showing the autoresonant evolution (solid lines) of $(\Phi+\pi)(mod\ 2\pi)-\pi$ (Fig. 7a) and $I$ (Fig. 7b) for the same parameters and initial conditions as in Fig. 2, but for longer times when the braking of the autoresonance is achieved. The dashed lines in the figures represent $\tilde{\Phi}$ and $\tilde{I}$ respectively. One can see that, as expected, the amplitude of the autoresonant phase oscillations decreases in time and that the autoresonance continues at $t \approx 1300$, in an excellent agreement with $t_b \approx 1278$ from (25) for this case.

### IV. CONCLUSIONS

We have studied the problem of trapping into resonance and subsequent autoresonant evolution of a nonlinear oscillator characterized by $x^4$-type potential and driven by a chirped frequency perturbation. We have shown that, under certain conditions, when the driving frequency increases and passes that of the unperturbed oscillator, the oscillator phase
increases in autoresonance. Generalization to gradually reduces the chirp rate as the oscillation amplitude
frequency chirp rate

\[ a_5 \sim G_n \]

shown in Fig. 3 in the energies takes place, until times scaling as
follows
locks to the drive, and, at later times, the oscillator frequency
independently of the initial phase of the oscillator. In contrast, for
these in the RHS of

case can be performed along the same lines as described in
In the general case, one deals with the characteristic function
where
[compare to (23) for the \( n = 2 \) case]

\[ \frac{1}{\nu^2} \frac{dv}{dt} < C_n \left[ \frac{1}{T^{n-3/2} r_n} + \frac{n \varepsilon \gamma_n/(n-1)}{c_n T^{3n-5/2} r_n} \right], \]

where \( C_n = (a/2)(n+1)^2 \left[ 2n(n-1)c_n \right]^{-3/2}(\varepsilon \gamma_n)^{-1/2} \). In addition to (27) one must also satisfy the condition similar to

\[ a(\varepsilon \gamma_n R_n I^2 n) < 1, \]

where \( R_n = [2n(n-1)/(n+1)^2] c_n I^{2n-1} + nr_c^2 \varepsilon \gamma_2 I^{-nr_n-1} \).

This condition guarantees the existence of a slow quasi-steady state in the system during the advanced autoresonance stage. As in the \( n = 2 \) case, (28) is not satisfied for all times, but is broken at [compare to (25)]

\[ t_b = \left[ \frac{2n(n-1)c_n \varepsilon \gamma_n}{\alpha(n+1)^2} \right]^{n+1} \left[ \frac{n+1}{\varepsilon \gamma_n(n-1)} \right]. \]

Equation (29) shows that, in the limit of large \( n \), for \( a \) fixed and small enough, the autoresonance lasts for longer times for larger values of \( n \).

In addition to these applications, we plan to extend our theory to the problem of sub- and higher harmonic autoresonance, as well as to higher dimensional dynamical systems of a similar type. Another interesting direction for future studies is the autoresonance in driven nonlinear wave systems which can be associated with the dynamical problem in \( \chi^2 \)-type potentials. One such example is the driven nonlinear Schrödinger equation with higher order nonlinearity [11]

\[ i \psi_t + \psi_{xx} + |\psi|^{2n} \psi = f. \]

The time independent limit of this equation for real \( \psi \) corresponds to the dynamical problem considered in this work.

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