A Non-Renormalization Theorem in Gapped QFT

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Our goal

To show that in 3D massive QFT,
the parity odd part of the 2-point function of the energy momentum tensor, $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$, is one-loop exact.

Outline

1. QED$_3$ and the Coleman-Hill theorem
2. A re-derivation of the Coleman-Hill theorem
3. A generalization to the energy momentum tensor

1. QED$_3$ and the Coleman Hill theorem

3D QED

We shall discuss QED_3, the class of theories given by

$$\mathcal{L}_0 = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}}$$

where $\mathcal{L}_{\text{matter}}$ is massive and

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \kappa \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

The equations of motion for A^μ are

$$\Box A^\mu - \left(\kappa e^2\right) \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = 0,$$

where κe^2 acts as a mass term.

The gauge field propagator

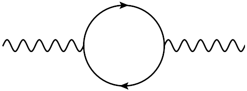
Consider the gauge field 2-point function

$$\langle A_\mu A_\nu \rangle = \Pi_1(p)\left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right) + \Pi_2(p)\,\epsilon_{\mu\nu\lambda} p^\lambda .$$

At zero momentum,

$$\Pi_2(0) = \kappa + \sum_{\text{Fermions}} -\!\!O\!\!- + \ldots$$

$-\!\!O\!\!-$ is the one loop graph



Each charged Fermion in the theory shifts κ by $q^2/4\pi$.

The Coleman-Hill theorem

Coleman and Hill showed that there can be no corrections above one loop. Their proof can be summarized as follows:

- A Feynman graph containing an uncharged particle cannot contribute to $\Pi_2(0)$, the parity odd part of $\langle A_\mu A_\nu \rangle$.
- A Feynman graph with three or more loops is equivalent to several graphs, each containing an uncharged particle.

As everything couples to gravity, there is no "uncharged particle!"

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This proof cannot be generalized to $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$!

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2. A re-derivation of the Coleman-Hill theorem

A re-derivation of the CH theorem

Instead of A_μ, we shall study j^μ, a (global) $U(1)$ current.
In massive theories, the parameterization of $\langle j_\mu(p) j_\nu(-p) \rangle$ is

$$a\delta_{\mu\nu} + \delta\kappa\, \epsilon_{\mu\nu\rho} p^\rho + O\left(p^2\right).$$

Note that only O(momentum) terms in the perturbative expansion can contribute to $\delta\kappa$ at zero momentum; we shall now see that no such terms exist.

A re-derivation of the CH theorem

Consider a generic QFT described by an action $\mathbf{S} = \int \mathcal{L}$, where

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \lambda_i O_i$$

and O_i are scalar operators.

The perturbative expansion reads

$$\langle j_\mu(p) j_\nu(-p) \rangle = \langle j_\mu(p) j_\nu(-p) \rangle_0 - \sum_i \lambda_i \langle j_\mu(p) j_\nu(-p) O_i(0) \rangle_0$$
$$+ \sum_{ij} \frac{1}{2} \lambda_i \lambda_j \langle j_\mu(p) j_\nu(-p) O_i(0) O_j(0) \rangle_0 + \ldots .$$

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A re-derivation of the CH theorem

In gapped theories, there are no infrared singularities, and so $\langle j_\mu(p) j_\nu(-p) O_1(0) \dots O_n(0) \rangle$ is well defined as the limit

$$\lim_{k_i \to 0} \langle j_\mu(p) j_\nu(q) O_1(k_1) \dots O_n(k_n) \rangle .$$

We will take this limit in two steps, $k_{i \neq 1} \to 0$ followed by $k_1 \to 0$. Consider the most general tensor structure of

$$\langle j_\mu(p) j_\nu(q) O_1(k_1) O_2(0) \dots O_n(0) \rangle .$$

- The insertion of $O_1(k_1)$ allows p and q to be independent.
- The insertions at zero momentum do not impose or relax any constraints on the tensor structure.

A re-derivation of the CH theorem

Consequently, the parameterization of

$$\langle j_\mu(p) j_\nu(q) O_1(k_1) O_2(0) \ldots O_n(0) \rangle$$

does not depend on the number of insertions at zero momentum - so let's study the 3-point function

$$\langle j_\mu(p) j_\nu(q) O(k_1) \rangle .$$

We can now take the limit $k_1 \to 0$: if $O(\mathrm{momentum})$ terms in $\langle j_\mu j_\nu O \rangle$ are for some reason forbidden, they must be absent from the rest of the perturbative corrections as well.

A reminder: Ward identities

When a theory is invariant under a continuous global transformation

$$\frac{\delta}{\delta\epsilon(x)} S' + \partial^{\mu} j_{\mu}(x) = 0.$$

If the symmetry is not anomalous, correlation functions are independent of the variation. In particular,

$$\frac{\delta}{\delta\epsilon(x)} \langle j_{\nu}(y) O(z) \rangle' = \frac{\delta}{\delta\epsilon(x)} \int_{\Phi} e^{-S'} j'_{\nu}(y) O'(z) = 0,$$

and so

$$\frac{\partial}{\partial x_{\mu}} \langle j_{\mu}(x) j_{\nu}(y) O(z) \rangle = -\langle \frac{\delta}{\delta\epsilon(x)} j'_{\nu}(y) O(z) \rangle - \langle j_{\nu}(y) \frac{\delta}{\delta\epsilon(x)} O'(z) \rangle.$$

A re-derivation of the CH theorem

The Ward identity for the $U(1)$ symmetry is just

$$p^\mu \langle j_\mu(p) j_\nu(q) O(k_1) \rangle = 0.$$

The parameterization of $\langle j_\mu(p) j_\nu(q) O(k_1) \rangle$ is given by

$$a' \delta_{\mu\nu} + b\, \epsilon_{\mu\nu\rho} (p^\rho - q^\rho) + O\left(\text{momentum}^2\right).$$

Both a' and b must vanish to satisfy the Ward identity.

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Both $a^{\text{'}}$ and b must vanish to satisfy the Ward identity.

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Another look 

2-point function:

$$\langle j_\mu(p) j_\nu(-p)\rangle = \ldots + \delta\kappa \, \epsilon_{\mu\nu\rho} p^\rho + \ldots$$

$$p^\mu \langle j_\mu(p) j_\nu(q) O(k_1)\rangle = 0 \implies \delta\kappa \ \text{ arbitrary}$$

3-point function:

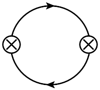
$$\langle j_\mu(p) j_\nu(q) O(k_1)\rangle = \ldots + b \, \epsilon_{\mu\nu\rho} \, (p^\rho - q^\rho) + \ldots$$

$$p^\mu \langle j_\mu(p) j_\nu(q) O(k_1)\rangle = 0 = -b \epsilon_{\mu\nu\rho} p^\mu q^\rho \implies b = 0$$

Remarks

1. What is so special about the one loop graph?

The only contribution to $\delta\kappa$, comes from $\langle j_\mu(p) j_\nu(q)\rangle_0$.
Since the current (in the free theory) is quadratic in the fields, $\langle j_\mu(p) j_\nu(q)\rangle_0$ corresponds to a one loop graph:



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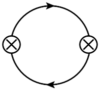

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Remarks

2. Why does the Ward identity forbid $O(\mathrm{momentum})$ terms in the tensor structure of $\langle j_\mu j_\nu O \rangle$?

Couple the global $U(1)$ current to a background gauge field a_μ, and the deformation O_i to a background source J_i.
We can then define

$$\langle j_\mu j_\nu O \rangle \equiv \frac{\delta}{\delta a^\mu} \frac{\delta}{\delta a^\nu} \frac{\delta}{\delta J} W[a, J_i] \Big|_{a=0, J_i=0}.$$

$O(\mathrm{momentum})$ terms in $\langle j_\mu j_\nu O \rangle$ correspond to terms in $W[a, J_i]$ with 2 $a's$, 1 J and only one derivative. There is one such term:

$$\int d^3x J \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho,$$

and it is ***NOT*** gauge invariant!

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3. Generalizing to the energy momentum tensor

Generalizing to $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$

The parameterization of $\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p) \rangle$ is

$$... + \delta\kappa_g \Big(\big(\epsilon_{\mu\rho\lambda} p^\lambda (p_\nu p_\sigma - p^2 \delta_{\nu\sigma}) + (\mu \leftrightarrow \nu) \big) + \rho \leftrightarrow \sigma) \Big) + O\left(p^4\right)$$

where $\delta\kappa_g$ is the shift in the gravitational Chern-Simons coefficiant.

The main difference is the Ward identity:

$$p^\mu \langle T_{\mu\nu}(p) T_{\rho\sigma}(q) O(-p-q) \rangle \neq 0,$$

as everything couples to gravity!

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Generalizing to $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$

The conservation of the energy-momentum tensor

$$\frac{\delta}{\delta \epsilon^{\mu}(x)} S' + \partial^{\nu} T_{\mu\nu}(x) = 0$$

is due to the invariance under the Poincaré group action

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}.$$

Under this transformation, the fields vary by a Lie derivative with respect to ϵ:

$$\Phi' = \Phi + L_{\epsilon}\Phi.$$

Generalizing to $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$

The variation of a scalar field ϕ is

$$\delta\phi = (\epsilon \cdot \partial)\phi,$$

and so the Ward identity for $\langle T_{\mu\nu} O \rangle$ reads

$$p^{\mu} \langle T_{\mu\nu}(p) O(q) \rangle = (p+q)_{\nu} \langle O(p+q) \rangle = 0.$$

Since $T_{\mu\nu}$ is symmetric, the parameterization of $\langle T_{\mu\nu}(p) O(-p) \rangle$ must be proportional to

$$p_{\mu} p_{\nu} - p^{2} \delta_{\mu\nu} + O\left(p^{4}\right).$$

Generalizing to $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$

The Ward identity for $\langle T_{\mu\nu} T_{\rho\sigma} O \rangle$ reads

$$p^\mu \langle T_{\mu\nu}(p) T_{\rho\sigma}(q) O(-p-q) \rangle \sim \mathrm{momentum} \times \langle TO \rangle .$$

Therefore, the only momentum^3 term in $\langle T_{\mu\nu}(p) T_{\rho\sigma}(q) O(k_1) \rangle$:

$$\mathrm{momentum}^2 \times \epsilon_{\mu\rho\lambda} \left(p^\lambda - q^\lambda \right) ,$$

cannot satisfy the Ward identity!

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there are no corrections to $\delta\kappa_g$!

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Remarks

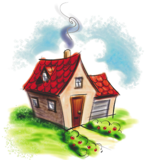
As in the $U(1)$ case, the non-renormalization of $\delta\kappa_g$ can be traced back to the properties of the generating functional as $T_{\mu\nu}$ couples to a background metric:

$$W[a_\mu, J] \ni \int d^3x\,(U(1)\text{ Chern-Simons})\cdot J \qquad \text{not gauge invariant}$$

$$\wr$$

$$W[g_{\mu\nu}, J] \ni \int d^3x\,(\text{g' Chern-Simons})\cdot J \qquad \text{not diff' invariant}$$

Take home message



Given two operators, A and B: if

$$\langle A(p) B(-p) \rangle$$

has a certian property, which is absent from the most general tensor structure of

$$\langle A(p) B(q) O(-p-q) \rangle$$

for an arbitrary scalar O -

that property is not renormalized!